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Coordinate Geometry and Vector Analysis

SEMESTER - V

MATHEMATICS

Block 1

KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

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CONTENTS

UNIT 1: The Straight Line **Pages: 5-28**

Slope of a Line, Equation of a Straight Line in Various Standard Forms, Some More Results

UNIT 2: Transformation of Co-ordinate Axes **Pages: 29-36**

Translation of Axes, Rotation of the Axes About the Origin

UNIT 3: Pair of Straight Lines **Pages: 37-58**

Pair of Straight Lines, Pair of Straight Lines Through the Origin, Condition for the most General Equation of the Second Degree to Represent a Pair of Straight Lines, Pair of Intersecting Straight Lines and Pair of Parallel Lines

UNIT 4: Circle **Pages: 59-84**

Standard Equation of a Circle, General Equation of a Circle, Equation of a Circle when End Points of its Diameter is Given, Tangent and Normal, Some Important Propositions

UNIT 5: Conic Section **Pages: 85-122**

Parabola and its Standard Equation, Ellipse and its Standard Equation, Hyperbola and its Standard Equation, General Equation of the Second Degree

UNIT 6: Introduction of Coordinate in Three Dimension **Pages: 123-148**

Rectangular Catesian Coordinates in Three Dimensions, Change of Origin (Without Changing the Direction of Axes), Distance Between Two Points, Section Formula, Direction Cosines of a Line, Direction Ratios of a Line

UNIT 7: The Plane **Pages: 149-170**

Equation of a Plane,Various forms of the Equation of a Plane

UNIT 8: The Straight Line **Pages: 171-196**

The Straight Line, The Shortest Distance (S.D.) Between Two Lines

COURSE INTRODUCTION

'Coordinate Geometry and Vector analysis' is the fifth course of the programme B. A Mathematics (pass). The course is divided into two blocks. The first block consists of eight units and second block consists of seven units.

Block 1 deals with the concept of straight lines and pair of straight line. Concept of circle and conic sections will be discussed in this block. Concept of coordinate geometry in three dimension will also introduced in this block.

Block 2 deals with the basic concept of sphere, cone and cylinder. Vectors and product of vectors will be introduced in this block. Concept of vector calculus will also covered in this block.

BLOCK INTRODUCTION

This is the first block of the course "Coordinate Geometry and Vector Analysis". The block consists of eight units.

In Unit 1, the basics of the straight line are introduced. In this unit, We will discuss different types of straight line.

In Unit 2, We will discuss the transformation of coordinate axes.

In Unit 3, discusses about pair of straight lines.

In Unit 4, covers the concept of circle.

In Unit 5, introduces fundamental concepts of conic sections.

In Unit 6, deals with the concept of three dimensional geometry.

In Unit 7, We will discuss about the plane.

In Unit 8, We will discuss about the straight line.

While going through a unit, you will notice some along-side boxes, which have been included to help you know some of the difficult, unseen terms. Again, we have included some relevant concepts in "LET US KNOW'' along with the text. And, at the end of each section, you will get "CHECK YOUR PROGRESS" questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with "ANSWERS TO CHECK YOUR PPROGRESS" given at the end of each unit.

UNIT 1: THE STRAIGHT LINE

UNIT STUCTURE

1.1 Learning Objectives

1.2 Introduction

1.3 Slope of a Line

1.4 Equation of a Straight Line in Various Standard Forms

1.5 Some More Results

1.6 Let Us Sum Up

1.7 Further Reading

1.8 Answers to Check Your Progress

1.9 Model Questions

1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know slope of a line
- find various form of the equation of the lines.

1.2 INTRODUCTION

In this unit, we will discuss slope of a line, angle between two lines and various forms of equations of a line. We will also discuss general equation of a line.

1.3 SLOPE OF A LINE

a) **Definition of the slope of a line and its formula:** If a straight line makes an angle θ with the positive direction of the x-axis (measured in the anticlockwise sense), then $\tan\theta$ is defined to be the slope (or gradient) of the line and is usually denoted by m.

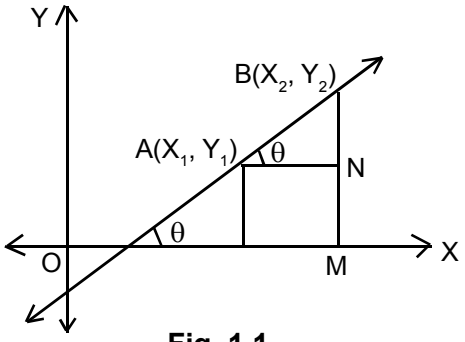


Fig. 1.1

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points on the line, Draw AL, BM $\perp$ to x-axis and AN $\perp$ BM. Then from Δ ABN,

$$\tan\theta = \frac{NB}{AN}$$

But, $NB = MB - MN = MB - LA = y_2 - y_1$

and $AN = LM = OM - OL = x_2 - x_1$

$$\therefore m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Thus the slope of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

As a special case if we consider a line joining origin (0, 0) to the point (x, y), we have,

$$m = \frac{y - 0}{x - 0} = \frac{y}{x}$$

All parallel lines make the same angle with the positive direction of x-axis and they have the same slope.

The x-axis makes angle 0 with itself and so its slope is 0, Hence any line parallel to the x-axis has slope 0.

The y-axis makes an angle of 90^0 with x-axis and its slope m = tan 90^0 is undefined; however, in the limiting sense, we sometimes say that slope of y-axis is infinite. This remark also applies to all lines parallel to y-axis.

b) Angle between two lines with given slope: Suppose that two lines make angles θ and θ' with the positive direction of the x-axis and ϕ is the angle between them.

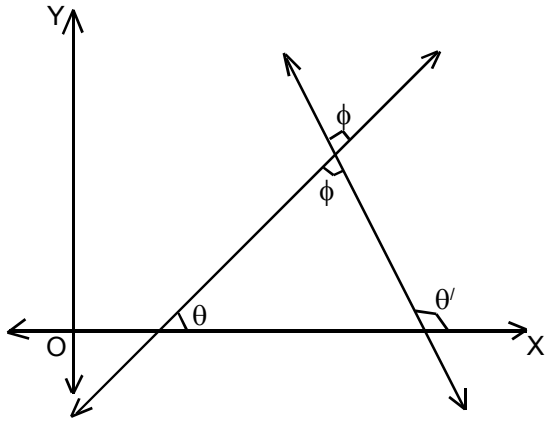


Fig. 1.2

Then, from the above figure,

$\phi = \theta' - \theta \Rightarrow \tan = \tan(\theta' - \theta) = \dfrac{\tan\theta' - \tan\theta}{1 + \tan\theta\tan\theta'}$

If m and m′ be the slopes of the above two lines, we get,

$\tan\phi = \dfrac{m' - m}{1 + mm'}$

This is the result corresponding to the above figure whose $\theta' > \theta$ i.e. m′ > m. In case $\theta > \theta'$ i.e m > m′.

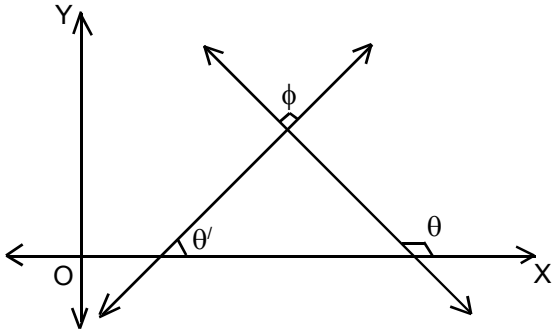


Fig. 1.3

as in the above figure, we shall get,

$\phi = \theta' - \theta \Rightarrow \tan\phi = \dfrac{m - m'}{1 + mm'}$

So, we ultimately have, $\tan\phi = \dfrac{m \sim m'}{1 + mm'}$

If the value of tanϕ is positive, ϕ is the acute angle between the two lines, if tanϕ is negative, ϕ is the obtuse angle between the two lines. The acute angle ϕ between the two lines is always given by the formula,

$$\tan\phi = \left|\frac{m' - m}{1 + mm'}\right| = \left|\frac{m - m'}{1 + mm'}\right|$$

Corollaries:

i) The two lines are parallel if $\phi = 0$ i.e. if $\tan\phi = 0$ i.e. if $m = m'$.

ii) The two lines are perpendicular if $\phi = 90^0$. i.e if $\cot\phi = 0$

i.e. if $\frac{1 + mm'}{m - m'} = 0$ i.e. if $mm' = -1$.

Illustrative Examples:

Example 1: What is the slope of the line joining origin to the point $A\left(1, -\sqrt{3}\right)$? What is the inclination of the line OA with the x-axis?

Solution: Slope of the line joining O(0, 0) to $A\left(1, -\sqrt{3}\right)$

$$= \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

If θ be the required inclination of OA with the x-axis, we have,

$\tan\theta = -\sqrt{3}$

$\Rightarrow \tan\theta = \tan 60^0 = \tan(180^0 - 60^0) = \tan 120^0$

$\Rightarrow \theta = 120^0$

Example 2: Given the points A(–1, 2), B(3, 4), C(0, –1) and D(5, 1), find the slopes of the lines AB and CD. Also find the acute angle between them.

Solution: m = slope of the line AB = $\frac{4-2}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$ and

m' = slope of the line CD = $\frac{1-(-1)}{5-0} = \frac{2}{5}$

If ϕ be the required acute angle between AB and CD, we have,

$$\tan\phi = \left|\frac{m - m'}{1 + mm'}\right| = \left|\frac{\frac{1}{2} - \frac{2}{5}}{1 + \frac{1}{2}\cdot\frac{2}{5}}\right| = \left|\frac{\frac{1}{10}}{\frac{6}{5}}\right| = \frac{1}{12}$$

$\therefore \phi = \tan^{-1}\frac{1}{12}$.

Example 3: Find the gradient of a line perpendicular to the line passing through the points (1, –3) and (–2, 4).

Solution: The gradient 'm' of the line passing through the points (1, –3) and (–2, 4) is given by,

$$m = \frac{4-(-3)}{-2-1} = -\frac{7}{3}$$

If 'm/' be the gradient of a line perpendicular to the above line,

we have $mm' = -1$

$$\Rightarrow \left(-\frac{7}{3}\right)m' = -1$$

$$\Rightarrow m' = \frac{3}{7}$$

$\therefore$ The required gradient is $\frac{3}{7}$.

Example 4: Using slopes find the value of x for which the points A(5, 1), B(1, –1) and C(x, 4) are collinear.

Solution: As the pts. A(5, 1), B (1, –1) and C(x, 4) are collinear, we have slope of AB = slope of AC

$$\Rightarrow \frac{-1-1}{1-5} = \frac{4-1}{x-5}$$

$$\Rightarrow \frac{1}{2} = \frac{3}{x-5}$$

$$\Rightarrow x - 5 = 6$$

$$\Rightarrow x = 1$$

Example 5: If the angle between two lines is 45^0 and the slope of one of them is $\frac{1}{3}$, find the slope of the other line.

Solution: Let the slope of the other line be m. The angle between the two lines having slopes $\frac{1}{3}$ and m is 45^0. Hence we have,

$$\left|\frac{m-\frac{1}{3}}{1+m.\frac{1}{3}}\right| = \tan 45^0 \Rightarrow \left|\frac{3m-1}{3+m}\right| = 1$$

$$\Rightarrow \frac{3m-1}{3+m} = \pm 1$$

$\Rightarrow \frac{3m-1}{3+m} = 1$ or, $\frac{3m-1}{3+m} = -1$

$\Rightarrow 3m - 1 = 3 + m$ or, $3m - 1 = -3 - m$

$\Rightarrow 2m = 4$ or, $4m = -2$

$\Rightarrow m = 2$ or, $m = -\frac{1}{2}$

$\therefore$ The required slope of the other line is 2 or, $-\frac{1}{2}$.

1.4 EQUATION OF A STRAIGHT LINE IN VARIOUS STANDARD FORMS

i) Point slope form: To find the equation of a straight line which passes through a given point A (x_1, y_1) and whose slope is m.

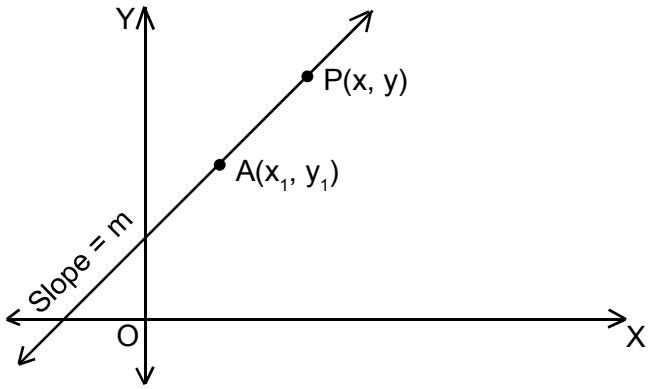


Fig. 1.4

Let P(x, y) be any point on the straight line. Now $A(x_1, y_1)$ and P(x, y) are two points on the line and so its slope m is given by

$$m = \frac{y - y_1}{x - x_1}$$

i.e., $y - y_1 = m(x - x_1)$

As this equation is satisfied by the coordinates (x, y) of each point P on the line, this is the equation of the line.

Thus the equation of the line passing through the point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.

In case the line pases through the origin (0, 0) and has slope m, its equation becomes y = mx.

ii) Two point form: To find the equation of a straight line which passes through two given points (x_1, y_1) and (x_2, y_2).

Let m be the slope of the line. As the line passes through the two given pts (x_1, y_1) and (x_2, y_2), we have, $m = \frac{y_2 - y_1}{x_2 - x_1}$

As the line has slope m and it passes through the point (x_1, y_1), the required equation of the line is, $y - y_1 = m(x - x_1)$

$$\text{i.e., } y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

which can be also be expressed as, $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$

iii) Slope intercept form: To find the equation of a straight line which has slope 'm' and which cuts off an intercept 'c' on y-axis.

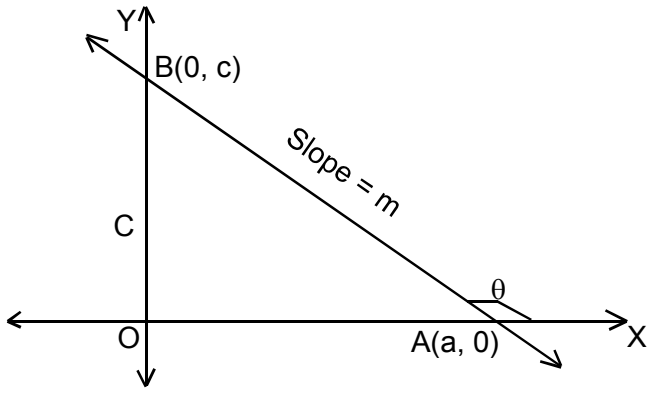


Fig. 1.5

Let AB be the line having slope m and cutting off an intercept OB = c on the y-axis. Then clearly B has coordinates (0, c). Thus the line AB has slope m and it passes through the point B(0. c). Hence the equation of the line AB is, $y - c = m(x - 0)$

$$\text{i.e., } y = mx + c.$$

In case the line passes through O (0, 0), c = 0 and the equation of the line becomes y = mx.

iv) Intercept form: To find the equation of a line cutting off intercepts a & b on the coordinate axes.

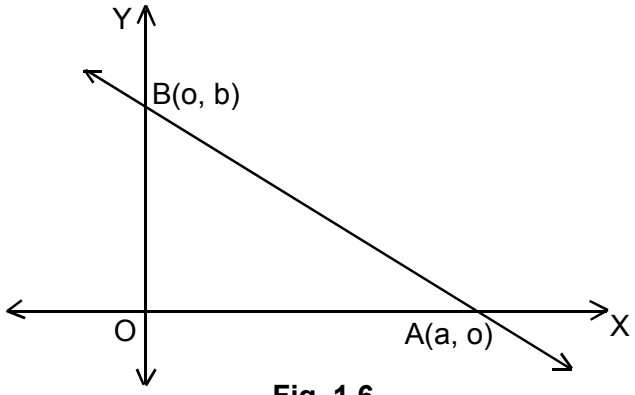


Fig. 1.6

Let AB be the straight line cutting off intercept OA = a on the x-axis and the intercept OB = b on the y-axis. Clearly the line passes through the points A(a, o) and B(o, b) and hence the equation of the line is

$$\frac{x-a}{0-a} = \frac{y-0}{b-0}$$

i.e., $b(x-a) = -ay$

i.e., $bx + ay = ab$

i.e., $\frac{bx}{ab} + \frac{ay}{ab} = \frac{ab}{ab}$

i.e., $\frac{x}{a} + \frac{y}{b} = 1$

Hence the equation of the line cutting off intercapts a and b on the axes of x and y respectively is $\frac{x}{a} + \frac{y}{b} = 1$.

v) Normal (or perpendicular) form: To find the equation of a straight line in terms of the length of the normal (perpendicular) on it from the origin and the angle thus this normal makes (with the positive direction of) x-axis (measured in the anticlockwise sense).

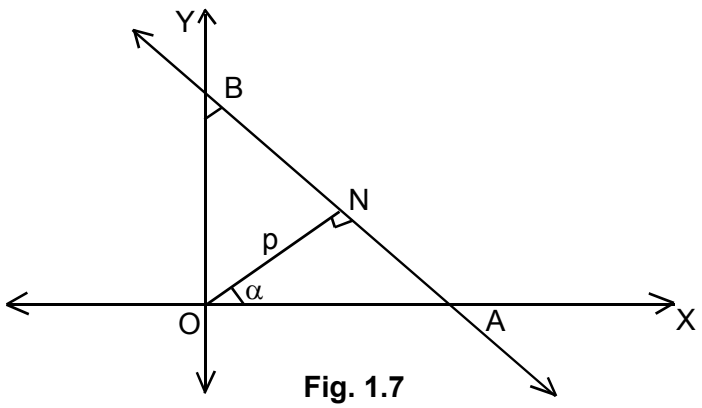


Fig. 1.7

Let ON be the normal from the origin to the straight line under consideration. Here ON = p and $\angle XON = \alpha$ are given; to find the equation of the line. Let the line meet x-axis at A and y-axis at B.

Then, $\frac{OA}{ON} = \sec\alpha \Rightarrow OA = p\sec\alpha$

and $\frac{OB}{ON} = \text{cosec}\alpha \Rightarrow OB = p\text{cosec}\alpha$

Now the line cuts off intercept OA = $p\sec\alpha$ on x-axis and intercept OB = $p\text{cosec}\alpha$ on y-axis; hence the equation of line is,

$$\frac{x}{p\sec\alpha} + \frac{y}{p\,\text{cosec}\alpha} = 1$$

i.e, $x\cos\alpha + y\sin\alpha = p$

1.5 SOME MORE RESULTS

The learners are advised to be aware of the following results so that they can apply them when necessary.

i) Every first degree equation in x and y represents a straight line. The most general equation of a straight line is Ax + By + C = O where A, B & C are arbitrary constants where at least one of A & B is non-zero. This equation can be reduced to various standard form.

Ax + By + C = O ... ...(1)

(1) can be expressed as, $y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$, if $B \neq 0$.

So, if $B \neq 0$, (1) represents a straight line whose gradient is $-\frac{A}{B}$ and intercept on y-axis is $-\frac{C}{B}$.

In case B = 0, (1) becomes Ax + C = 0, i.e., $x = -\frac{C}{A}$

Which represents a straight line parallel to y-axis.

(1) can also be expressed as Ax + By = – C

i.e., $\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1$ ($A \neq 0$, $B \neq 0$)

Which is the equation of the line (1) in intercept form.

To reduce (1) to the normal form, we write it as,

$Ax + By = -C$, if, $C < 0$ (i.e., if, $-C > 0$)

or, $(-A)x + (-B)y = C$, if, $C > 0$

In either case dividing both sides by $\sqrt{A^2 + B^2}$, we get,

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}} \text{ (in case, } C < 0)$$

$$\frac{-A}{\sqrt{A^2+B^2}}x + \frac{-B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}} \text{ (in case, } C > 0)$$

as the requried normal form of the line represented by $Ax + By + C = 0$.

ii) The two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are parallel if, $\frac{A_1}{A_2} = \frac{B_1}{B_2}$ and they are perpendicular if, $A_1A_2 + B_1B_2 = 0$.

iii) Any line perpendicular to the line $Ax + By + C = 0$ is given by $Bx - Ay + K = 0$, where K is an arbitrary constant.

iv) Any line parallel to the line $Ax + By + C = 0$ is given by $Ax + By + K = 0$, where K is an arbitrary constant.

v) The equation of any straight line passing through the point of intersection of the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ is $A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0$, where k is an arbitrary constant.

vi) The length of the perpendicular drawn from the point (x_1, y_1) to the line, $Ax + By + C = 0$ is, $\left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}}\right|$.

vii) The equations of the bisectors of the angles between the two lines $A_1x + B_1y + C_1 = 0$ & $A_2x + B_2y + C_2 = 0$ are,

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm\frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}.$$

If, $C_1 > 0$ and $C_2 > 0$, + sign will give the bisector of the angle which contains the origin.

viii) The three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ will be concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

ILLUSTRATIVE EXAMPLES:

Example 1: i) Find the equation of the straight line passing through the points (1, –2) and (2, 3).

ii) What intercepts are made by it on the axes?

iii) Obtain the normal form of the equation of the line.

Solution: i) The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is, $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$

In this case the two points are $(x_1, y_1) \equiv (1, -2)$ and $(x_2, y_2) \equiv (2, 3)$.

Hence the required equation of the line is $\frac{x-1}{2-1} = \frac{y-(-2)}{3-(-2)}$

i.e., $\frac{x-1}{1} = \frac{y+2}{5}$

i.e., $5x - 5 = y + 2$

i.e., $5x - y = 7$

ii) The equation of the line is $5x - y = 7$, i.e., $\frac{5x - y}{7} = 1$,

i.e., $\frac{5x}{7} - \frac{y}{7} = 1$, i.e., $\frac{x}{7/5} + \frac{y}{-7} = 1$

Which is in the intercept form $\frac{x}{a} + \frac{y}{b} = 1$.

$\therefore$ Intercept on x-axis = a = $\frac{7}{5}$ and Intercept on y-axis = b = –7.

iii) The equation of the line is $5x - y = 7$

[Note that the constant is on the RHS & positive]

Dividing both sides by $\sqrt{5^2 + (-1)^2}$ i.e., $\sqrt{26}$, we get,

$$\frac{5}{\sqrt{26}}x + \frac{-1}{\sqrt{26}}y = \frac{7}{\sqrt{26}}$$

Which is the required normal form of equation of the line.

Example 2: i) Find the equation of the line which makes an angle of $\frac{\pi}{4}$ with the positive direction of x-axis and cuts off intercept 2 on the y-axis.

ii) Reduce the equation of the line to the normal form. What angle does the normal from the origin to the line make with the x-aixs? What is the distance of the line from the origin?

Solution: i) The slope of the line is, m = $\tan\frac{\pi}{4}$ = 1

The intercept made by the line on the y-axis is c = 2.

$\therefore$ The equation of the line is y = mx + c

i.e., y = x + 2; i.e., x – y + 2 = 0

ii) The equation of the line is x – y + 2 = 0

i.e., x – y = – 2; i.e., –x + y = 2

[Note that the constant is taken on the R.H.S. and is made positive]

Dividing both sides by $\sqrt{(-1)^2+1^2}$ i.e., $\sqrt{2}$

we get, $\left(-\frac{1}{\sqrt{2}}\right)x+\frac{1}{\sqrt{2}}y=\sqrt{2}$

Which is the equation of the line in the normal form.

$x\cos\alpha + y\sin\alpha = p\cos\alpha$

Here $\cos\alpha = -\frac{1}{\sqrt{2}}$, $\sin\alpha = \frac{1}{\sqrt{2}}$, p = $\sqrt{2}$

We get, $\frac{\sin\alpha}{\cos\alpha}$ = –1

$\Rightarrow \tan\alpha = -1$

$\Rightarrow \alpha = \frac{3\pi}{4}$ or, $\frac{7\pi}{4}$

As $\cos\alpha < 0$ and $\sin\alpha > 0$, α lies in the 2nd quadrant, So, $\alpha = \frac{3\pi}{4}$

$\therefore$ The normal to the line makes an angle of $\frac{3\pi}{4}$ (i.e. 135^0)

with x-axis.

Distance of the line from the origin = p = $\sqrt{2}$.

Example 3: Find the equations of the lines through the point (3, –1) which are parallel and perpendicular to the line 3x – 4y = 1.

Solutions: Any line parallel to the line 3x – 4y = 1 is,

3x – 4y = k ... ...(1)

If it passes through the point (3, –1), then we must have,

$3.3 – 4.(–1) = k \Rightarrow k = 13$

Then (1) becomes $3x – 4y = 13$

Which is the required equation of the line through (3, –1) and parallel to the line $3x – 4y = 1$.

Any line perpendicular to the given line $3x – 4y = 1$ is,

$4x + 3y = k'$ ... ...(2)

If it passes through the point (3,-1), then we must have,

$4.3 + 3(–1) = k' \Rightarrow k' = 9$

Then (2) becomes $4x + 3y = 9$ which is the reqd. equation of the line through the point (3, –1) and perpendicular to the line $3x – 4y = 1$.

Example 4: Calculate the length of the perpendicular from the point (–3, –1) to the line $\frac{x}{2} - \frac{y}{3}. = –1$.

Solution: The line is, $\frac{x}{2} - \frac{y}{3} = -1$

i.e., $3x – 2y = –6$

i.e., $3x – 2y + 6 = 0$

Length of the perpendicular from the point (–3, –1) to the above line

$$= \left| \frac{3.(-3) - 2.(-1) + 6}{\sqrt{3^2 + (-2)^2}} \right|$$

$$= \left| \frac{-1}{\sqrt{13}} \right|$$

$$= \frac{1}{\sqrt{13}}$$

Example 5: If p is the length of the perpendicular from the origin to the line $\frac{x}{a} + \frac{y}{b} = 1$, prove that, $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$

Solution: $p = \frac{1}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}}$

$$\Rightarrow \frac{1}{p} = \sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{a^2}+\frac{1}{b^2}$$

Example 6: Find the distance of the point of intersection of the lines $2x - y - 1 = 0$ and $x - 3y + 2 = 0$ from the point $(2, -5)$.

Solution: The given lines are,

$2x - y - 1 = 0$ ... ...(1)

$x - 3y + 2 = 0$ ... ...(2)

Solving the equations (1) and (2) we get,

$$\frac{x}{-2-3} = \frac{y}{-1-4} = \frac{1}{-6+1}$$

$$\Rightarrow \frac{x}{-5} = \frac{y}{-5} = \frac{1}{-5}$$

$\Rightarrow x = 1, y = 1$

$\therefore$ The point of intersection of the given lines (1) and (2) is (1, 1).

Hence the required distance of this point of intersection (1, 1) from the point $(2, -5) = \sqrt{(1-2)^2+(1+5)^2} = \sqrt{1+36} = \sqrt{37}$

Example 7: Find the equation to the straight line which passes through the intersection of the lines $x + 2y + 3 = 3$ and $3x + 4y + 7 = 0$ and is perpendicular to the straight line $y - x = 8$

Solution: Equation of any line passing through the point of intersection of the lines $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ is,

$x + 2y + 3 + k(3x + 4y + 7) = 0$

i.e., $(1 + 3k)x + (2 + 4k)y + (3 + 7k) = 0$ ... ...(1)

It will be perpendicular to the straight line $y - x = 8$,

i.e., $x - y + 8 = 0$ if, $(1 + 3k) + (-1)(2 + 4k) = 0$

(condition: $a_1a_2 + b_1b_2 = 0$)

i.e., $1 + 3k - 2 - 4k = 0$; i.e., $-k - 1 = 0$; i.e., $k = -1$

Putting $k = -1$ in (1) we get, $-2x - 2y - 4 = 0$

i.e., $x + y + 2 = 0$,

Which is the equation of the straight line required.

Example 8: For what value of m will the three lines

y = 3x –1, 2y = x + 3 and 3y = mx + 4 be concurrent?

Solution: The given lines are,

y = 3x – 1, 2y = x + 3, 3y = mx + 4

i.e., 3x – y – 1 = 0, x – 2y + 3 = 0, mx – 3y + 4 = 0.

For these three lines to be concurrent we should have,

$$\begin{vmatrix} 3 & -1 & -1 \\ 1 & -2 & 3 \\ m & -3 & 4 \end{vmatrix} = 0$$

$\Rightarrow$ 3(–8 + 9) + 1(4 – 3m) + (–1)(–3 + 2m) = 0

$\Rightarrow$ 3 + 4 – 3m + 3 – 2m = 0

$\Rightarrow$ –5m = –10

$\Rightarrow$ m = 2

Example 9: Find the equation of the line which passes through the point (4, –7) and is 8 units away from the origin.

Solution: Any line through (4, –7) is y – (–7) = m(x – 4)

i.e., mx – y – (4m + 7) = 0 ... ...(1)

In order that this line may be 8 units away from the origin, the length of the perpendicular to this line from the origin should be 8; thus,

$$\frac{|-(4m+7)|}{\sqrt{m^2+(-1)^2}} = 8$$

$\Rightarrow |4m+7| = 8\sqrt{m^2+1}$

$\Rightarrow (4m+7)^2 = 64(m^2+1)$

(on squaring, noting that $|p|^2 = (\pm p)^2 = p^2$)

$\Rightarrow$ $16m^2 + 56m + 49 = 64m^2 + 64$

$\Rightarrow$ $48m^2 – 56m + 15 = 0$

$$\Rightarrow m = \frac{-(-56) \pm \sqrt{(-56)^2 - 4.48.15}}{2 \times 48}$$

$$= \frac{56 \pm \sqrt{8^2 \times 7^2 - 8^2 \times 3 \times 15}}{2 \times 48}$$

$$= \frac{56 \pm 8\sqrt{49-45}}{2\times 48}$$

$$= \frac{56 \pm 8\times 2}{2\times 48}$$

$$= \frac{72}{2\times 48} \text{ or, } \frac{40}{2\times 48}$$

$$\Rightarrow m = \frac{3}{4} \text{ or, } \frac{5}{12}$$

For $m = \frac{3}{4}$, (1) becomes, $\frac{3}{4}x - y - 10 = 0$, i.e., $3x - 4y - 40 = 0$

For $m = \frac{5}{12}$, (1) becomes, $\frac{5}{12}x - \frac{26}{3}y - = 0$, i.e., $5x - 12y - 104 = 0$

$\therefore$ The required equation of the line is,

$3x - 4y - 40 = 0$ or $5x - 12y - 104 = 0$

Remark: The reason for double answers is that there are two lines through (4, –7) which are 8 units away from the origin.

Example 10: Find the distance between the two parallel lines $y = mx + c$ and $y = mx + d$ $(c > d)$.

Solution: The distance between two parallel lines is equal to the length of the perpendicular drawn from any point on one of the lines to the other.

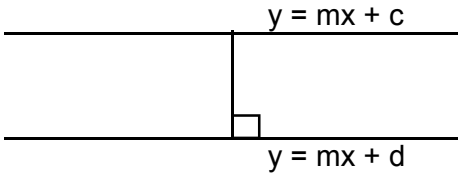


Fig. 1.8

Putting $x = 0$ in $y = mx + c$ we get $y = c$, thus $(0, c)$ is a point on the line $y = mx + c$. The length of the perpendicular from $(0, c)$ to the other line $y = mx + d$ i.e. $mx - y + d = 0$ is,

$$\frac{|m.0 - c + d|}{\sqrt{m^2 + (-1)^2}} = \frac{|d - c|}{\sqrt{m^2 + 1}} = \frac{c - d}{\sqrt{m^2 + 1}} \quad (c > d)$$

$\therefore$ Reqd. distance between the two parallel lines $= \frac{c - d}{\sqrt{m^2 + 1}}$

Example 11: Find the equations of the bisectors of the angles between the lines 3x – 4y – 5 = 0 and 12x + 5y + 1 = 0.

Solution: The given lines have equations,

3x – 4y – 5 = 0 and 12x+ 5y +1 = 0

i.e., –3x + 4y + 5 = 0 and 12x + 5y + 1 = 0

(on making the constant terms in both the equations positive).

The equation of the bisector of the angle between the lines which contains the origin is,

$$\frac{-3x+4y+5}{\sqrt{(-3)^2+4^2}} = \frac{12x+5y+1}{\sqrt{12^2+5^2}}$$

i.e., 5(12x + 5y +1) = 13(–3x + 4y + 5)

i.e., 99x – 27y – 60 = 0,

i.e., 33x – 9y – 20 = 0

The equation of the other bisector is,

$$\frac{-3x+4y+5}{\sqrt{(-3)^2+4^2}} = -\frac{12x+5y+1}{\sqrt{12^2+5^2}}$$

i.e., –39x + 52y + 65 = –60x – 25y – 5

i.e., 21x +77y + 70 = 0,

i.e., 3x + 11y +10 = 0



CHECK YOUR PROGRESS

Q.1: Find the equation of the line which passes through the point (3, –1) and makes equal intercepts on the axes.

Q.2: What is the slope of the line 3x + 4y = 1 ? Find the equations of the line through the point (2, –3) which are parallel and perpendicular to it.

Q.3: Find the point of intersection of the lines 3x – 4y – 6 = 0 and 2x + y + 7 = 0.

Q.4: Find the angle between the lines 5x – y = 9 and x + 6y = 8.

Q.5: Find the distance between the parallel lines 2x + 3y = 6 and 6x +9y = 5.

Q.6: Show that the lines 5x + 3y = 7, 3x – 4y = 10 and x + 2y = 0 meet at a point. What is that point?

Q.7: Find the equation of the line through the point of intersection of the lines 5x + 3y = 7 and 3x – 4y = 10 and parallel to the line 3x – 2y = 0.

Q.8: Find the equations of the bisectors of the angles between the lines 3x – 4y + 7 = 0 and 12x – 5y – 8 = 0.



1.6 LET US SUM UP

- If a straight line passes through the two points (x_1, y_1) and (x_2, y_2), then its slope (or gradient) m is given by $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$
- The slope m of a line joining origin to the point (x, y) is given by $m = \frac{y}{x}$
- The angle ϕ between two straight lines having slopes m and m' is given by $\tan\phi = \frac{m \sim m'}{1 + mm'}$
- Two lines having slopes m and m' are parallel if $m = m'$.
- Two lines having slopes m and m' are perpendicular if $mm' = -1$
- The equation of the straight line passing through the point (x_1, y_1) and having 'm' as its slope is $y - y_1 = m(x - x_1)$ (Point - slope Form)
- The equation of the line passing through the two points (x_1, y_1) and (x_2, y_2) is $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$ (Two point form)
- The equation of the line which has slope 'm' and which cuts off intercept 'c' on y-axis is y = mx + c. (slope-intercept form)
- The equation of the line cutting off intercepts a and b on the axes of x and y respectively is $\frac{x}{a} + \frac{y}{b} = 1$. (Intercept Form)
- If p is the length of the normal from the origin to a line and the normal makes an angle α with x-axis, then the equation of the line is, $x\cos\alpha + y\sin\alpha = p$. (Normal Form)



1.7 FURTHER READING

1) Jain, P. K. and Ahmhad, K. *Textbook of Analytical Geometry of Three Dimensions*. New Delhi: New Age International (P) Ltd.
2) Kishan, Hari. *Coordinate Geometry of Two dimensions*. New Delhi: Atlantic Publishers & Distributors.
3) Pal, Sat and Pal, Harbans. *A Textbook of Two Dimensional Geometry*. New Age International Publishers.
4) Sharma, G. C. and Jain, Madhu. *Coordinate Geometry (2D and 3D)*. New Delhi: Galgotia Publications Pvt Ltd.
5) Sharma, A. K. *Coordinate Geometry (2D)*. New Delhi: Discovery Publishing House.



1.8 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: The line makes equal intercepts on the axes, say a and a. Then its equation is, $\frac{x}{a}+\frac{y}{a}=1$

i.e., $x + y = a$

As this line passes through the point (3, – 1), we must have,

$3 + (–1) = a$

Whence $a = 2$

$\therefore$ The required equation of the line is $x + y = 2$

Ans. to Q. No. 2: The equation of the given line is,

$3x + 4y = 1$

i.e., $4y = –3x + 1$

i.e., $y = \left(-\frac{3}{4}\right)x+\frac{1}{4}$

Which is of the form $y = mx + c$.

$\therefore$ Slope of the line $= m = -\frac{3}{4}$

A line which is parallel to the given line has the same slope $m = -\frac{3}{4}$.

So, the equation of the line through the point (2, –3) which is parallel to the given line is, $y - (-3) = \left(-\frac{3}{4}\right)(x-2)$

i.e., $4(y + 3) = -3(x - 2)$

i.e., $3x + 4y + 6 = 0$

The slope, m', of a line perpendicular to the given line is given by

$mm' = -1$

$\Rightarrow m' = -\frac{1}{m} = -\frac{1}{-\frac{3}{4}} = \frac{4}{3}$

So the equation of the line through the point (2, –3) which is perpendicular to the given line is,

$y - (-3) = \left(-\frac{3}{4}\right)(x-2)$

i.e., $3y + 9 = 4x - 8$

i.e., $4x - 3y - 17 = 0$

Ans. to Q. No. 3: The coordinates of the point of intersection of the line.

$3x - 4y - 6 = 0$ and $2x + y + 7 = 0$

is obtained by solving the above two equations. By the rule of cross multication we have,

$$\frac{x}{-28+6} = \frac{y}{-12-21} = \frac{1}{3+8}$$

$\Rightarrow \frac{x}{-22} = \frac{y}{-33} = \frac{1}{11}$

$\Rightarrow x = -2,\ y = -3.$

$\therefore$ The required point of intersection is (–2, –3).

Ans. to Q. No. 4: The line $5x - y = 9$ i.e., $y = 5x - 9$ has slope $m = 5$ and the line $x + 6y = 8$, i.e., $y = -\frac{1}{6}x + \frac{8}{6}$ has slope $m' = -\frac{1}{6}$. If ϕ be the acute angle between the two lines, we have,

$$\tan\phi = \left|\frac{m - m'}{1 + mm'}\right| = \left|\frac{5 + \frac{1}{6}}{1 - 5.\frac{1}{6}}\right|$$

$$= \left|\frac{31/6}{1/6}\right| = 31$$

$\Rightarrow \phi = \tan^{-1}31$

Ans. to Q. No. 5: The parallel lines are,

$2x + 3y = 6$; i.e., $2x + 3y - 6 = 0$ ... ...(1)

and $6x + 9y = 5$; i.e., $6x + 9y - 5 = 0$ ... ...(2)

Putting $x = 0$ in (1) we get $y = 2$. Thus (0, 2) is a point on the line (1).

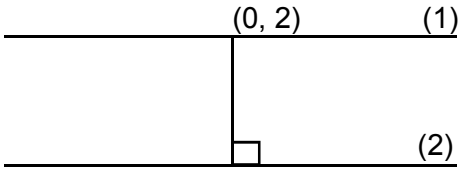


Fig. 1.11

The required distance between the parallel lines (1) and (2)

= the length of the perpendicular from the point (0, 2) on the line (1) to the line (2)

$$= \left|\frac{6.0 + 9.2 - 5}{\sqrt{6^2 + 9^2}}\right| = \frac{13}{\sqrt{36 + 81}} = \frac{13}{3\sqrt{13}} = \sqrt{\frac{13}{3}}$$

Ans. to Q. No. 6: $5x + 3y = 7$ ... ...(1)

$3x - 4y = 10$ ... ...(2)

$x + 2y = 0$ ... ...(3)

From (1) and (2), by Cramer's rule,

$$\frac{x}{\begin{vmatrix} 7 & 3 \\ 10 & -4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 5 & 7 \\ 3 & 10 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 5 & 3 \\ 3 & -4 \end{vmatrix}}$$

$$\Rightarrow \frac{x}{-28 - 30} = \frac{y}{50 - 21} = \frac{1}{-20 - 9}$$

$$\Rightarrow \frac{x}{-58} = \frac{y}{29} = \frac{1}{-29}$$

$\Rightarrow x = 2,\ y = -1.$

$\therefore$ The point of intersection of the lines (1) and (2) is (2, –1).

When x = 2 & y = –1,

L.H.S. of (3) = 2 + 2. (–1) = 2 – 2 = 0

$\therefore$ The line (3) passes through the point (2, –1).

Hence the three lines (1), (2) and (3) meet at a point as all of them pass through the point (2, –1). That point is (2, –1).

Ans. to Q. No. 7: The equation of any line passing through the point of intersection of the lines 5x + 3y = 7 and 3x – 4y = 10, i.e., of the lines 5x + 3y – 7 = 0 and 3x – 4y – 10 = 0 is,

$$5x + 3y - 7 + \lambda(3x - 4y - 10) = 0$$

i.e., $(5 + 3\lambda)x + (3 - 4\lambda)y - (7 + 10\lambda) = 0$ ... ...(1)

This line will be parallel to the line 3x – 2y = 0 if,

$$\frac{5 + 3\lambda}{3} = \frac{3 - 4\lambda}{-2}$$

$\Rightarrow -10 - 6\lambda = 9 - 12\lambda$

$\Rightarrow 6\lambda = 19$

$\Rightarrow \lambda = \frac{19}{6}$

On putting $\lambda = \frac{19}{6}$ in (1) we get,

$$\left(5 + \frac{57}{6}\right)x + \left(3 - \frac{76}{6}\right)y - \left(7 + \frac{190}{6}\right) = 0$$

i.e., $87x - 58y - 232 = 0$

i.e., $29(3x - 2y - 8) = 0$

i.e., $3x - 2y - 8 = 0$

Which is the equation of the line as required.

Ans. to Q. No. 8: The equations of the bisectors of the angles between the lines

3x – 4y + 7 = 0 and 12x – 5y – 8 = 0 are,

$$\frac{3x - 4y + 7}{\sqrt{3^2 + (-4)^2}} = \pm\frac{12x - 5y - 8}{\sqrt{12^2 + (-5)^2}}$$

i.e., $\frac{3x - 4y + 7}{5} = \pm\frac{12x - 5y - 8}{13}$

Corresponding to +sign, we get,

$39x - 52y + 91 = 60x - 25y - 40$

i.e., $21x + 27y - 131 = 0$,

Corresponding to –sign, we get,

$39x - 52y + 91 = -(60x - 25y - 40)$

i.e., $99x - 77y + 51 = 0$.

Hence the required equations of the bisectors of the angles between the two lines are $21x + 27y - 131 = 0$ and $99x - 77y + 51 = 0$.



1.9 MODEL QUESTIONS

Q.1: Find the equation of the straight line which cuts an intercept 5 from y-axis and makes an angle of 30^0 with x-axis.

Q.2: A(3, 3); B(–3, –5) and C(–1, 6) are the vertices of ΔABC. Find the slopes of the medians.

Q.3: Find the equation of a straight line which passes through the point (4, 6) and makes intercepts on the axes equal in magnitude but opposite in sign.

Q.4: A straight line passes through the point (4, 5) such that the portion intercepted by the axes bisected at that point. Find the equation of the straight line.

Q.5: A straight line moves so that the sum of the reciprocals of its intercepts on two perpendicular lines is constant. Show that it always passes through a fixed point.

Q.6: Find the equation of a straight line such that the length of the perpendicular from the origin upon it is $4\sqrt{2}$ units and the angle which this perpendicular makes with x-axis is $\tan^{-1}1$.

Q.7: In what ratio is the line segment joining (2, 3) and (–1, 4) divided by the line $x + y + 1 = 0$?

Q.8: Find the distance of the point (2, 3) from the line $2x - 3y + 9 = 0$ measured along a line making an angle of 45^0 with x-axis.

Q.9: Find the angle between the lines

i) $2x - y + 3 = 0$ and $x + y = 0$

ii) $x + 2y + 9 = 0$ and $3x + y - 7 = 0$.

Q.10: Find the equations of two straight lines through the point (4,5) which make an angle of 45^0 with the line $2x - y + 7 = 0$.

Q.11: Find the equation of the line passing through the intersection of $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ and perpendicular to the line $y - x = 9$.

Q.12: Find the lengths of the perpendiculars from the vertices on the opposite sides of the triangle whose vertices are (0, 0), (1, –1), (3, 2).

Q.13: Find the distance between the parallel lines

i) $9x + 40y - 20 = 0$, $9x + 40y + 103 = 0$

ii) $5x + 12y - 7 = 0$, $5x + 12y + 45 = 0$.

Q.14: Find the equations of the lines drawn through the point (0, 1) on which the perpendiculars dropped from the point (2, 2) are each of unit point.

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UNIT 2: TRANSFORMATION OF COORDINATE AXES

UNIT STUCTURE

2.1 Learning Objectives

2.2 Introduction

2.3 Transformation of Co-ordinate Axes

2.4 Let Us Sum Up

2.5 Further Reading

2.6 Answers to check your progress

2.7 Model Questions

2.1 LEARNING OBJECTIVES

After going through this unit, you will able to:

- know transformation of coordinate axes.

2.2 INTRODUCTION

In this unit, we will discuss transformation of coordinate axes.

2.3 TRANSFORMATION OF CO-ORDINATE AXES

a) Change of origin without changing the directions of the axes (Translation of Axes):

Let (h, k) be the coordinates of O' referred to the coordinate axes OX and OY. Let us now take a pair of new coordinate axes $O'X'$ and $O'Y'$ parallel to the old pair, O' being the new origin.

Let P be any point in the plane whose coordinates are say (x, y) referred to the old axes and (x', y') referred to the new axes. Draw $PN \perp OX$ intersecting $O'X'$ perpendicularly at N' and let $O'Y'$ intersect OX perpendicularly at K. Then clearly

$$x = ON = OK + KN = OK + O'N' = h + x'$$

and $y = NP = NN' + N'P = k + y'$

Thus we get the following formulae of transformation if origin is

shifted to the point (h, k) without changing the direction of axes:

$x = x' + h$ and $y = y' + k$.

b) Change in the directions of the axes without changing the origin (i.e. Rotation of the axes about the origin):

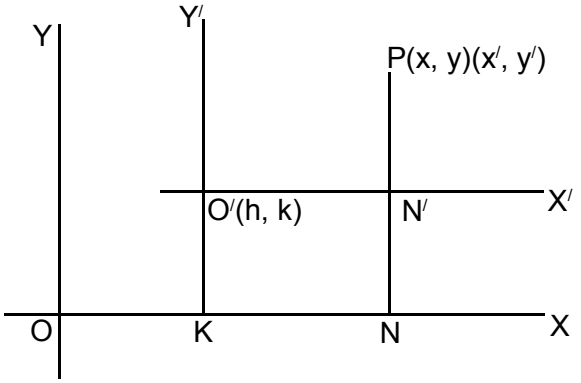


Fig. 2.1

Suppose that the two axes OX and OY are rotated through an angle θ whereby they occupy new positions OX' and OY' respectively.

Let (x, y) be the coordinates of any point P in the plane referred to the original axes OX, OY and (x', y') be its coordinates referred to new axes OX', OY'; O is the origin in either system. Then from the above figure which is self-explanatory, we get,

$$x = ON = OK - NK = OK - TM$$
$$= OM\cos\theta - MP\sin\theta = x'\cos\theta - y'\sin\theta$$

and $$y = NP = NT + TP = KM + TP$$
$$= OM\sin\theta + MP\cos\theta = x'\sin\theta + y'\cos\theta.$$

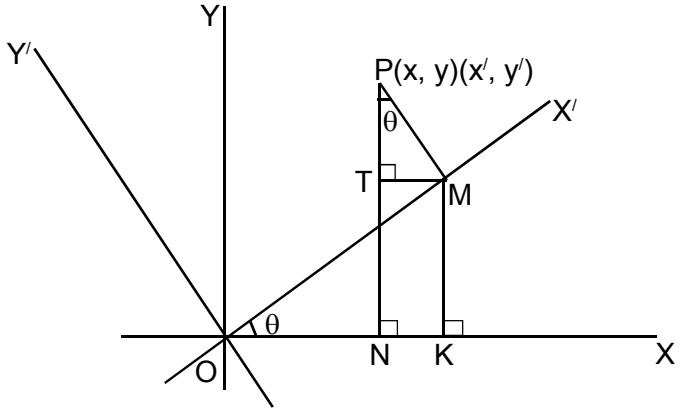


Fig. 2.2

Thus in case of rotation of the axes through an angle θ, without changing the origin, the transformation formulae are:

$x = x'\cos\theta - y'\sin\theta$

and $y = x'\sin\theta + y'\cos\theta$.

ILLUSTRATIVE EXAMPLE:

Example 1: Find the transformed equation of the curve $y^2 = 4x - 2y - 7$ when the origin is transferred to the point (2, –1).

Solution: When origin is transferred to the point (2, –1), the new coordinates (x', y') are given by $x = x' + 2$, $y = y' - 1$.

Hence the equation $y^2 = 4x - 2y - 7$ becomes,

$(y' - 1)^2 = 4(x' + 2) - 2(y' - 1) - 7$

i.e., $y'^2 - 2y' + 1 = 4x' + 8 - 2y' + 2 - 7$

i.e., $y'^2 = 4x' + 2$

If now the new coordinates are denoted by (x, y), the transformed equation is $y^2 = 4x + 2$.

Example 2: Find the transformed equation $x^2 + 3y^2 + 4x + 18y + 30 = 0$, when the origin is transferred to the point (–2, –3).

Solution: When origin is transferred to the point (–2, –3), the new coordinates (x', y') are given by $x = x' - 2$, $y = y' - 3$.

Hence the equation $x^2 + 3y^2 + 4x + 18y + 30 = 0$ becomes,

$(x' - 2)^2 + 3(y' - 3)^2 + 4(x' - 2) + 18(y' - 3) + 30 = 0$

i.e., $x'^2 + 3y'^2 = 1$

If now the new coordinates are denoted by (x, y), the transformed equation is $x^2 + 3y^2 = 1$.

Example 3: Transform the equation $x^2 - y^2 = a^2$ to axes inclined at 45^0 to the original axes.

Solution: If (x', y') be the new coordinates of the point whose original coordinates were (x, y), we have,

$$x = x'\cos 45^0 - y'\sin 45^0 = \frac{x' + y'}{\sqrt{2}}$$

and $$y = x'\sin 45^0 + y'\cos 45^0 = \frac{x' + y'}{\sqrt{2}}$$

So, the equation $x^2 - y^2 = a^2$ becomes,

$$\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - \left(\frac{x' + y'}{\sqrt{2}}\right)^2 = a^2$$

i.e., $-2x'y' = a^2$, i.e., $2x'y' + a^2 = 0$

Hence the transformed equation is $2xy + a^2 = 0$

Where now, as usual, (x, y) are the new coordinates.

Example 4: Transform the equation $x^2 - y^2 + 2\sqrt{3}xy = 2a^2$ to axes inclined at 30^0 to the original axes.

Solution: If (x', y') be the new coordinates of the point whose original coordinates were (x, y), we have,

$$x = x'\cos 30^0 - y'\sin 30^0 = x'\frac{\sqrt{3}}{2} - y'.\frac{1}{2} = \frac{\sqrt{3}x' - y'}{2}$$

and $$x = x'\sin 30^0 + y'\cos 30^0 = x'\frac{1}{2} + y'.\frac{\sqrt{3}}{2} = \frac{x' + \sqrt{3}y'}{2}$$

So, the equation $x^2 - y^2 + 2\sqrt{3}xy = 2a^2$ becomes,

$$\Rightarrow \left(\frac{\sqrt{3}x' - y'}{2}\right)^2 - \left(\frac{x' + \sqrt{3}y'}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) = 2a^2$$

$$\Rightarrow \left(\frac{3x'^2 - 2\sqrt{3}x'y' + y'^2}{4}\right) - \left(\frac{x'^2 + 2\sqrt{3}x'y' + 3y'^2}{4}\right) + 2\sqrt{3}\left(\frac{\sqrt{3}x'^2 + 3x'y' - x'y' - \sqrt{3}y'^2}{4}\right) = 2a^2$$

$$\Rightarrow \frac{3x'^2 - 2\sqrt{3}x'y' + y'^2 - x'^2 - 2\sqrt{3}x'y' - 3y'^2 + 2\sqrt{3}\left(\sqrt{3}x'^2 + 2x'y' - \sqrt{3}y'^2\right)}{4} = 2a^2$$

$$\Rightarrow 3x'^2 - 2\sqrt{3}x'y' + y'^2 - x'^2 - 2\sqrt{3}x'y' - 3y'^2 + 6x'^2 + 4\sqrt{3}x'y' - 6y'^2 = 8a^2$$

$$\Rightarrow 8x'^2 - 8y'^2 = 8a^2$$

$$\Rightarrow x'^2 - y'^2 = a^2$$

Hence the transformed equation is $x^2 - y^2 = a^2$.

Where now, as usual, (x, y) are the new coordinates.

Example 5: Show that by transformation from one set of rectangular axes to another with the same origin a first degree homogeneous expression ax + by changes to a first degree homogeneous expression. Also show that under this transformation $a^2 + b^2$ is an invariant.

Solution: Using the transformation formulae,

$x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ we get,

$ax + by = a(x'\cos\theta - y'\sin\theta) + b(x'\sin\theta + y'\cos\theta)$

$= (a\cos\theta + b\sin\theta)x' + (b\cos\theta - a\sin\theta)y'$

$= a'x' + b'y'$

Where, $a' = a\cos\theta + b\sin\theta$

and $b' = b\cos\theta - a\sin\theta$.

As $a'x' + b'y'$ is a first degree homogeneous expression in x' and y', we see that the first degree homogeneous expression $ax + by$ transforms to a first degree homogeneous expression $a'x' + b'y'$. Now,

$a'^2 + b'^2 = (a\cos\theta + b\sin\theta)^2 + (b\cos\theta - a\sin\theta)^2$

$= (a^2\cos\theta + b^2\sin^2\theta + 2ab\sin\theta\cos\theta) +$

$(b^2\cos^2\theta + a^2\sin^2\theta - 2ab\sin\theta\cos\theta$

$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$

$= a^2 + b^2$

This shows that for the expression $ax + by$, the quantity $a^2 + b^2$ is an invariant under rectangular transformation with the same origin.



CHECK YOUR PROGRESS

Q.1: Transform the equations given below to parallel axes through the new origin (2, –1):

i) $5x + 3y = 3$ ii) $x^2 + y^2 = 6$

Q.2: Transform the equations $3x^2 + 2xy + 3y^2 = 1$, when the axes are turned through an angle of $\frac{\pi}{4}$ through the same origin.

Q.3: If by change of axes, without change of origin, the expression $ax^2 + 2hxy + by^2$ becomes $ax'^2 + 2hx'y' + by'^2$, prove that,

i) $a + b = a' + b'$ ii) $ab - h = a'b' - h'$

iii) $(a-b)^2 + 4h^2 = (a'-b')^2 + 4h'^2$



2.4 LET US SUM UP

- If origin is shifted to the point (h, k) and through it new axes parallel to the old axes are taken, transformation formulae for coordinates are, $x = x' + h$, $y = y' + k$; Where (x, y) are the old coordinates and (x', y') are the new coordinates of the same point.
- If, without changing the origin, the axes are rotated through are angle θ, then the transformation formulae are,
  $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$.



2.5 FURTHER READING

1) Jain, P. K. and Ahmhad, K. *Textbook of Analytical Geometry of Three Dimensions*. New Delhi: New Age International (P) Ltd.
2) Kishan, Hari. *Coordinate Geometry of Two dimensions*. New Delhi: Atlantic Publishers & Distributors.
3) Pal, Sat and Pal, Harbans. *A Textbook of Two Dimensional Geometry*. New Age International Publishers.
4) Sharma, G. C. and Jain, Madhu. *Coordinate Geometry (2D and 3D)*. New Delhi: Galgotia Publications Pvt Ltd.
5) Sharma, A. K. *Coordinate Geometry (2D)*. New Delhi: Discovery Publishing House.



2.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: The new origin is (2, –1), the new axes being parallel the original axes. To get the transformed equation we have to replace x by x + 2 and y by y – 1 ($\because x = x' + h$ and $y = y' + k$).

i) The equation $5x + 3y = 3$ transforms to $5(x+2) + 3(y–1) = 3$
i.e., $5x + 3y + 4 = 0$

ii) The equation $x^2 + y^2 = 6$ transformes to $(x+2)^2 + (y-1)^2 = 6$

i.e., $x^2 + y^2 + 4x - 2y - 1 = 0$.

Ans. to Q. No. 2: When the axes are turned through an angle $\theta = \frac{\pi}{4}$,

We have, $x = x'\cos\theta - y'\sin\theta = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = \frac{x' - y'}{\sqrt{2}}$

and $y = x'\sin\theta + y'\cos\theta = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = \frac{x' - y'}{\sqrt{2}}$

Where (x', y') denote the new coordinates of the point whose old coordinates were (x, y).

Hence the equation, $3x^2 + 2xy + 3y^2 = 1$ transforms to:

$$3\left(\frac{x' - y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 3\left(\frac{x' + y'}{\sqrt{2}}\right)^2 = 1$$

i.e., $\frac{3}{2}\left[\left(x' - y'\right)^2 + \left(x' + y'\right)^2\right] + \left(x' - y'\right)\left(x' + y'\right) = 1$

i.e., $\frac{3}{2}.2\left(x'^2 + y'^2\right) + x'^2 - y'^2 = 1$

i.e., $4x'^2 + 2y'^2 = 1$

Denoting the new coordinates by (x, y), the transformed equation is

$4x^2 + 2y^2 = 1$

Ans. to Q. No. 3: Let the axes be turned through an angle θ. Using the transformation formulae

$x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

The expression $ax^2 + 2hxy + by^2$ becomes,

$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) +$

$b(x'\sin\theta + y'\cos\theta)^2 = a'x'^2 + 2h'x'y' + b'y'^2$

$\Rightarrow a(x'^2\cos^2\theta - 2x'y'\cos\theta\sin\theta + y'^2\sin^2\theta) +$

$2h(x'^2\cos\theta\sin\theta + x'y'\cos^2\theta - x'y'\sin^2\theta + y'^2\sin\theta\cos\theta) +$

$b(x'^2\sin^2\theta + 2x'y'\sin\theta\cos\theta + y'^2\cos^2\theta) = a'x'^2 + 2h'x'y' + b'y'^2$

Comparing both sides, we get,

$a' = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$

$b' = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$

$h' = h(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta$

Now,

i) $a' + b' = a(\cos^2\theta + \sin^2\theta) + b(\sin^2\theta + \cos^2\theta) = a + b$

ii) $2a' = a(2\cos^2\theta) + 2h(2\sin\theta\cos\theta) + b(2\sin^2\theta)$

$= a(1+\cos2\theta) + 2h\sin2\theta + b(1-\cos2\theta)$

$= (a+b) + [2h\sin2\theta + (a-b)\cos2\theta]$

Similarly, $2b' = (a+b) + [2h\sin2\theta + (a-b)\cos2\theta]$

and $2h' = 2h\cos2\theta - -(a-b)\sin2\theta$

$\therefore 4(a'b' - h'^2 = 2a'.2b' - (2h')^2$

$= [(a+b) + \{2h\sin2\theta + (a-b)\cos2\theta\}][(a+b) - \{2h\sin2\theta + (a-b)\cos2\theta\}]$

$-[2h\cos2\theta - (a-b\sin2\theta)]^2$

$= (a+b)^2 - [2h\sin2\theta + (a-b)\cos2\theta)]^2 - [2h\cos2\theta - (a-b)\sin2\theta]^2$

$= (a+b)^2 - 4h^2(\sin^2\theta + \cos^2\theta) - (a-b)^2(\cos^2 2\theta + \sin^2 2\theta)$

$= (a+b)^2 - 4h^2 - (a-b)^2$

$= 4ab - 4h^2$

$= 4(ab-h^2)$

$\therefore a'b' - h'^2 = 4(ab-h^2)$

iii) $(a-b)^2 + 4h^2 = (a+b)^2 - 4ab + 4h'^2$

$= (a+b)^2 - 4(ab-h^2)$

$= (a'+b')^2 - 4(a'b' - h'^2)$ using (i) and (ii)

$= (a' - b')^2 + 4h'^2$



2.7 MODEL QUESTIONS

Q.1: Find the transformed equation of the straight line $\frac{x}{a} + \frac{y}{b} = 1$, when the origin is transferred to the point (a, b).

Q.2: Transformed the equation $17x^2 - 16xy + 17y^2 = 225$ to axes inclined at 60^0 to the original axes.

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UNIT 3: PAIR OF STRAIGHT LINES

UNIT STUCTURE

3.1 Learning Objectives

3.2 Introduction

3.3 Pair of Straight Lines

3.3.1 Pair of Straight Lines Through the Origin

3.3.2 Angle Between the Pair of Lines Given by $ax^2 + 2hxy + by^2 = 0$

3.3.3 Bisectors of the Angle Between the Pair of Lines $ax^2 + 2hxy + by^2 = 0$

3.4 Condition for the Most General Equation of the Second Degree to Represent a Pair of Straight Lines

3.5 Pair of Intersecting Straight Lines and Pair of Parallel Lines

3.6 Let Us Sum Up

3.7 Further Reading

3.8 Answers to Check Your Progress

3.9 Model Questions

3.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know pair of straight lines
- find angle between the pair of lines
- find the bisectors of the angle between the pair of lines
- know pair of intersecting straight lines and pair of parallel straight lines.

3.2 INTRODUCTION

In unit 1, we have already discussed the straight line and its various forms. In this unit, we will discuss the pair of straight lines, angle between pair of straight lines and condition for representing a pair of straight lines. We will also discuss the pair of intersecting straight lines and pair of parallel lines in this unit.

3.3 PAIR OF STRAIGHT LINES

Consider the equation $x^2 + 2xy + y^2 - 4 = 0$, which is a 2nd degree equation in x and y. This equation can be expresed as, $(x + y)^2 - 4 = 0$

i.e., $(x + y + 2)(x + y - 2) = 0$

The given equation, $x^2 + 2xy + y^2 - 4 = 0$ ... ...(1)

is satisfied if either, $x + y + 2 = 0$ ... ...(2)

or, $x + y - 2 = 0$ ... ...(3)

Geometrically it means that any point which lies on the locus represented by either (2) or (3) lies on the locus represented by (1) and vice-versa. But (2) and (3), being 1st degree equations in x and y, represent straight lines. Hence the equation (1) represents a pair of straight lines whose separate equations are (2) and (3).

We see that a particular 2nd degree equation may represent a pair of straight lines.

3.3.1 Pair of Staight Lines Through the Origin

The general homogeneous equation of the 2nd degree in x and y is,

$ax^2 + 2hxy + by^2 = 0$ ... ...(1)

Now, $ax^2 + 2hxy + by^2 = 0$

i.e., $b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$ ... ...(2)

This is a quadratic equation in $\frac{y}{x}$. let its roots be m_1 and m_2.

Then we have, $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.

Hence (2) can be expressed as,

$$\left(\frac{y}{x}\right)^2 + \frac{2h}{b}\left(\frac{y}{x}\right) + \frac{a}{b}, = 0 \; b \neq 0$$

i.e., $\left(\frac{y}{x}\right)^2 - (m_1 + m_2)\left(\frac{y}{x}\right) + m_1m_2 = 0$

i.e., $\left(\frac{y}{x} - m_1\right) - \left(\frac{y}{x} - m_2\right) = 0$

i.e., $(y - m_1x)(y - m_2x) = 0$ ... ...(3)

(1) is equivalent to (3). Clearly (3) represents the pair of lines, $y - m_1x = 0$ and $y - m_2x = 0$ through the origin. Hence (1) represents a pair of straight lines through the origin whose individual equations are $y - m_1x = 0$ and $y - m_2x = 0$.

3.3.2 Angle Between the Pair of Lines Given by $ax^2 + 2hxy + by^2 = 0$

Let the separate equations of the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y = m_1x$ and $y = m_2x$. Then we have,

$m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.

If θ be the required angle between the two lines, we have,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$= \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$

$$= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4.\frac{a}{b}}}{1 + \frac{a}{b}}$$

$$= \frac{2\sqrt{h^2 - ab}}{a + b}$$

(1) If $h^2 = ab$, then $\tan\theta = 0$ i.e. $\theta = 0$, so that, the two lines coincide.

(2) If $a + b = 0$, then $\cot\theta = \frac{a + b}{2\sqrt{h^2 - ab}} = 0$, so that $\theta = \frac{\pi}{2}$, implying that the two lines are perpendicular.

3.3.3 Bisectors of the Angles Between the Pair of Lines $ax^2 + 2hxy + by^2 = 0$

Let the individual equations of a the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$

Where $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.

The equations of the pair of bisectors of the angles between the above two lines are, $\frac{y - m_1x}{\sqrt{1+m_1^2}} = \pm\frac{y - m_2x}{\sqrt{1+m_2^2}}$

These two equations can be combined into a single equation as,

$$\frac{(y - m_1x)^2}{1+m_1^2} = \frac{(y - m_2x)^2}{1+m_2^2}$$

i.e., $(1 + m^2_2)(y^2 + m^2_1x^2 - 2m_1xy) = (1 + m^2_1)(y^2 + m_2^2x^2 - 2m_2xy)$

i.e., $\{(1 + m_2^2)m^2_1 - (1 + m_1^2)m_2^2)\}x^2 - 2\{(1 + m_2^2)m_1 - (1 + m_1^2)m_2\}xy + \{(1 + m_2^2) - (1 + m_1^2)\}y^2 = 0$

i.e., $(m_1^2 - m^2_2)x^2 - 2\{(m_1 - m_2) - m_1m_2(m_1 - m_2)\}xy - (m_1^2 - m_2^2)y^2 = 0$

i.e., $(m_1^2 - m^2_2)(x^2 - y^2) = 2(m_1 - m_2)(1 - m_1m_2)xy$

i.e., $(m_1 + m_2)(x^2 - y^2) = 2(1 - m_1m_2)xy$

i.e, $-\frac{2h}{b}(x^2 - y^2) = 2\left(1 - \frac{a}{b}\right)xy = -\frac{2(a - b)xy}{b}$

i.e., $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

This is the required equation of the pair of bisectors.

ILLUSTRATIVE EXAMPLES:

Example 1: Form the equation which represents the pair of lines $4x + y = 0$ and $3x = 2y$.

Solution: The given lines are,

$4x + y = 0$ and $3x = 2y$

i.e., $4x + y = 0$ and $3x - 2y = 0$

The required equation which represents the above pair of lines is,

$(4x + y)(3x - 2y) = 0$

i.e., $12x^2 - 5xy - 2y^2 = 0$

Example 2: Identify the locus represented by $x^2 + xy - 6y^2 = 0$.

Solution: $x^2 + xy - 6y^2 = 0$ is a homogeneous equation of 2nd degree. Hence it represents a pair of straight lines through the origin. Now the given equation is,

$$x^2 + xy - 6y^2 = 0$$

i.e., $(x - 2y)(x + 3y) = 0$

So the separate equations of the above two lines are,

$x - 2y = 0$ and $x + 3y = 0$

Example 3: Find the angle between the lines represented by

$2x^2 - 7xy + 3y^2 = 0$.

Solution: Comparing the given equation $2x^2 - 7xy + 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$ we get $a = 2$, $h = -\frac{7}{2}$, $b = 3$. If θ be the angle between the lines represented by $2x^2 - 7xy + 3y^2 = 0$, we have,

$$\tan\theta = \frac{2\sqrt{\left(-\frac{7}{2}\right)^2 - 2.3}}{2+3} = \frac{2\sqrt{\frac{25}{4}}}{5} = 1$$

$\therefore \theta = 45^0$, which is the required angle.

Example 4: Find the value of k for which the two lines

$3x^2 + xy + ky^2 = 0$ are perpendicular to each other.

Solution: For the two lines $3x^2 + xy + ky^2 = 0$ to be perpendicular to each other then, we must have, $3 + k = 0 \Rightarrow k = -3$.

Example 5: Find the equation of the bisectors the angles between the pair of lines $5x^2 + xy - 6y^2 = 0$.

Solution: Comparing with $ax^2 + 2hxy + by^2 = 0$,

We have, $a = 5$, $h = \frac{1}{2}$, $b = -6$.

The reqd. equation of the bisectors is, $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

i.e., $\frac{x^2 - y^2}{5 + 6} = \frac{xy}{\frac{1}{2}}$

i.e., $x^2 - y^2 = 22xy$

i.e., $x^2 - 22xy - y^2 = 0$

Example 6: If the two pairs of lines. $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angles between the other pair, prove that $pq = -1$.

Solution: The equation representing the bisectors of the angle between the pair of lines $x^2 - 2pxy - y^2 = 0$ is,

$$\frac{x^2 - y^2}{1-(-1)} = \frac{xy}{-p}$$

i.e., $px^2 - py^2 = -2xy$

i.e., $px^2 + 2xy - py^2 = 0$ ... ...(1)

But it is given that these bisectors are represented by the equation,

$x^2 - 2qxy - y^2 = 0$ ... ...(2)

So, (1) and (2) must be identical. Hence we get,

$$\frac{p}{1} = \frac{2}{-2q} = \frac{-p}{-1}$$

$\Rightarrow pq = -1$.



CHECK YOUR PROGRESS

Q.1: Find the equation which represents the pair of lines $3x - y = 0$ and $x + 2y = 1$.

Q.2: Find the angle between the pair of lines represented by

i) $8x^2 + 10xy + 3y^2 = 0$ ii) $x^2 - 2xy\sec\alpha + y^2 = 0$

Q.3: Find the value of λ for which the two lines,

$3x^2 - 8xy + \lambda y^2 = 0$ are:

i) perpendicular to each other. ii) coincident.

Q.4: Find the equation of the bisectors of the angles between the pair of lines

i) $15x^2 - xy - 6y^2 = 0$ ii) $x^2 - 3xy + 2y^2 = 0$

Q.5: If the pair of lines $x^2 + 4xy - y^2 = 0$ bisects the angles between the pair of lines $x^2 + 2kxy - y^2 = 0$, then find the value of k.

3.4 CONDITION FOR THE MOST GENERAL EQUATION OF THE SECOND DEGREE TO REPRESENT A PAIR OF STRAIGHT LINES

The most general equation of the 2nd degree x and y is,

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ...(1)

Arranging (1) as a quadratic equation in x, we get,

$ax^2 + 2(hy + g)x + (by^2 + 2fy + c) = 0$

So that,

$$x = \frac{-(hy+g) \pm \sqrt{(hy+g)^2 - a(by^2+2fy+c)}}{a}$$

$$\text{i.e., } x = \frac{-(hy+g) \pm \sqrt{(h^2+ab)y^2 + 2(gh-af)y + (g^2-ac)}}{a} \quad (a \neq 0)$$

... ...(2)

(1) will represent a pair of straight lines if its L.H.S. can be expresed as the product of two linear factor in x and y. This will be possible if the expression under the radical sign in the R.H.S. of (2), viz.

$(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)$

is perfect square for which we must have,

$(gh - af)^2 - (h^2 - ab)(g^2 - ae) = 0$

$\Rightarrow (g^2h^2 - 2afgh + a^2f^2) - (g^2h^2 - abg^2 - ach^2 + a^2bc) = 0$

$\Rightarrow -2afgh - a^2bc + a^2f^2 + abg^2 + ach^2 = 0$

$\Rightarrow -a(abc + 2fgh - af^2 - bg^2 - ch^2) = 0$

$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\therefore$ The condition that (1) may represent a pair of straight lines is,

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$

Note:

i) As, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$, the condition obtained above may be expressed also in the form $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

ii) The above condition for pair of lines is obtained under the supposition $a \neq 0$. It holds even when a = 0.

ILLUSTRATIVE EXAMPLES:

Example 1: Does the equation $6x^2 - 12xy - 6x^2 + 14x + 8y + 4 = 0$ represents a pais of straight lines?

Solution: Comparing the given equation

$6x^2 - 12xy - 6y^2 + 14x + 8y + 4 = 0$ ... ...(1)

with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

$a = 6, h = -6, b = -6, g = 7, f = 4, c = 4.$

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2$

$= 6.(-6).4 + 2.4.7.(-6) - 6.4^2 - (-6).7^2 - 4.(-6)^2$

$= -144 - 336 - 96 + 294 - 144$

$= -720 + 294 = -426 \neq 0$

Hence (1) does not represent a pair of straight lines.

Example 2: Find the value of λ so that the equation $x^2 + 2\lambda xy + y^2 + 6x + 2y + 9 = 0$ may represent a pair of lines.

Solution: The equation, $x^2 + 2\lambda xy + y^2 + 6x + 2y + 9 = 0$ will represent a pair of lines if, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Here, $a = 1, h = \lambda, b = 1, g = 3, f = 1, c = 9$

i.e., $1.1.9 + 2.1.3\lambda - 1.1^2 - 1.3^2 - 9.\lambda^2 = 0$

i.e., $9 + 6\lambda - 1 - 9 - 9\lambda^2 = 0$

i.e., $-9\lambda^2 + 6\lambda - 1 = 0$

i.e., $9\lambda^2 - 6\lambda + 1 = 0$

i.e., $(3\lambda - 1)^2 = 0$

i.e., $3\lambda - 1 = 0$

i.e., $3\lambda = 1$

i.e., $\lambda = \frac{1}{3}$

3.5 PAIR OF INTERSECTING STRAIGHT LINES & PAIR OF PARALLEL LINES

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ...(1)

Suppose that, $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

So that, (1) represents a pair of st. lines. In this case

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \equiv (lx + my + n)(l'x + m'y + n')$

... ...(2)

the separate equations of the lines given by (1) being

$lx + my + n = 0$ ... ...(3)

and $l'x + m'y + n' = 0$ ... ...(4)

$(lm' - l'm \neq 0)$

Now (2) is equivalent to

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \equiv$

$ll'x^2 + (lm' + l'm)xy + mm'y^2 + (nl' + n'l)x + (nm' + m'n)y + nn';$

Comparing coefficients we have,

$ll' = a,\ mm' = b,\ nn' = c,\ lm' + l'm = 2h,\ nl' + n'l = 2g,\ mn' + m'n = 2f.$

If ϕ be the angle between the lines (3) and (4), we have,

$$\tan\phi = \frac{lm' - l'm}{ll' + mm'}$$

$$= \frac{\sqrt{(lm' + l'm)^2 - 4ll'mm'}}{ll' + mm'}$$

$$= \frac{\sqrt{(2h)^2 - 4ab}}{a + b}$$

$$= \frac{2\sqrt{h^2 - ab}}{a + b}$$

A) Case of Intersecting Straight Lines: If $h^2 \neq ab$, then $\tan\phi \neq 0$, i.e., $\phi \neq 0$, and hence the lines (3) and (4) are intersecting. Hence if $\Delta = 0$ and $h^2 \neq ab$, (1) represents a pair of intersecting straight lines.

Here a special case arise when $a + b = 0$. Then, $\phi = 90^0$ and the two lines intersect at right angles. Thus if $\Delta = 0$ and $a + b = 0$, (1) represents a pair of perpendicular straight lines.

B) Case of Parallel Straight Lines: If $h^2 = ab$, then $\tan\phi = 0$, so that, $\phi = 0$, and so the lines (3) and (4) are parallel. Hence if $\Delta = 0$ and $h^2 = ab$, (1) represents a pair of parallel st. lines (including the case of coincident lines)

ILLUSTRATIVE EXAMPLES:

Example 1: Prove that the equation $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ represents a pair of perpendicular lines.

Solution: Comparing the given equation,

$6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ ... ...(1)

with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we have,

$a = 6, h = -\frac{5}{2}, b = -6, g = 7, f = \frac{5}{2}, c = 4,$

$\therefore abc + 2fgh - af^2 - bh^2 - ch^2$

$= 6.(-6).4 + 2.\frac{5}{2}.7.\left(-\frac{5}{2}\right) - 6.\left(\frac{5}{2}\right)^2 - (-6).7^2 - 4.\left(-\frac{5}{2}\right)^2$

$= -144 - \frac{175}{2} - \frac{75}{2} + 294 - 25$

$= -144 - 125 + 269 = 0$

$\therefore$ (1) represents a pair of lines.

Moreover $a + b = 6 + (-6) = 0$

It follows that (1) represents a pair of perpendicular lines.

Example 2: Show that $4x^2 + 5xy + y^2 + 3y - 4 = 0$ represents a pair of lines and find the angle between them.

Solution: $4x^2 + 5xy + y^2 + 3y - 4 = 0$ ... ...(1)

Here, $a = 4, h = \frac{5}{2}, b = 1, g = 0, f = \frac{3}{2}, c = -4.$

Now, $abc + 2fgh - af^2 - bg^2 _ ch^2$

$= 4.1.(-4) + 2.\frac{3}{2}.0.\frac{5}{2} - 4.\left(\frac{3}{2}\right)^2 - 1.0^2 - (-4).\left(\frac{5}{2}\right)^2$

$= -16 - 9 + 25 = 0$

$\therefore$ (1) represents a pair of lines.

If ϕ be the angle between the lines given by (1), we have,

$$\tan\phi = \frac{2\sqrt{h^2 - ab}}{a + b} = \frac{2\sqrt{\left(\frac{5}{2}\right)^2 - 4.1}}{4 + 1} = \frac{2 \times \frac{3}{2}}{5} = \frac{3}{5}$$

$$\Rightarrow \phi = \tan^{-1}\frac{3}{5}$$

Example 3: Prove that the equation $2x^2 + xy - 6y^2 - 6x + 23y - 20 = 0$ represents a pair of straight lines. Find the coordinates of their point of intersection and the angle between them.

Solution: $2x^2 + xy - 6y^2 - 6x + 23y - 20 = 0$ ... ...(1)

$\Rightarrow 2x^2 + (y - 6)x + (23y - 20 - 6y^2) = 0$

$$\Rightarrow x = \frac{-(y-6) \pm \sqrt{(y-6)^2 - 4.2.(23y - 20 - 6y^2)}}{2.2}$$

$$= \frac{(-y+6) \pm \sqrt{(y^2 - 12y + 36) - (184y - 160 - 48y^2)}}{4}$$

$$= \frac{(-y+6) \pm \sqrt{(49y^2 - 196y + 196)}}{4}$$

$$= \frac{(-y+6) \pm 7\sqrt{y^2 - 4y + 4}}{4}$$

$$= \frac{(-y+6) \pm 7\sqrt{(y-2)^2}}{4}$$

$$\Rightarrow x = \frac{(-y+6) \pm 7(y-2)}{4}$$

Corresponding to +sign, we get,

$$x = \frac{(-y+6) + 7(y-2)}{4} = \frac{6y-8}{4} = \frac{3y-4}{2}$$

i.e., $2x - 3y + 4 = 0$;

Corresponding to –sign, we get,

$$x = \frac{(-y+6) - 7(y-2)}{4} = \frac{-8y+20}{4} = -2y + 5$$

i.e., $x + 2y - 5 = 0$

Hence the given equation (1) represents a pair of straight lines whose individual equations are

$2x - 3y + 4 = 0$ ... ...(2)

and $x + 2y - 5 = 0$ ... ...(3)

From (2) and (3) $\frac{x}{15-8} = \frac{y}{4+10} = \frac{1}{4+3}$

$$\Rightarrow \frac{x}{7} = \frac{y}{14} = \frac{1}{7}$$

$\Rightarrow x = 1, y = 2$

$\therefore$ The coordinates of the point of intersection of the two lines represented by (1) are (1, 2).

Slope of the line (2) is m = $\frac{2}{3}$, and Slope of the line (3) is m′ = $-\frac{1}{2}$

$\therefore$ Required angle between the two lines = $\tan^{-1}\frac{m-m'}{1+mm'}$

$$= \tan^{-1}\frac{\frac{2}{3}+\frac{1}{2}}{1+\frac{2}{3}.\left(-\frac{1}{2}\right)}$$

$$= \tan^{-1}\frac{\frac{7}{6}}{\frac{2}{3}} = \tan^{-1}\frac{7}{4}$$

Example 4: Show that the equation $9x^2 - 12xy + 4y^2 - 15x + 10y + 6 = 0$ represents a pair of parallel straight lines.

Solution: The given equation is,

$9x^2 - 12xy + 4y^2 - 15x + 10y + 6 = 0$ ... ...(1)

Here, in usual notations,

$a = 9, h = -6, b = 4, g = -\frac{15}{2}, f = 5, c = 6,$

Now, $abc + 2fgh - af^2 - bg^2 - ch^2$

$= 9.4.6 + 2.5.\left(-\frac{15}{2}\right).(-6) - 9.5^2 - 4.\left(-\frac{15}{2}\right)^2 - 6.(-6)^2$

$= 216 + 450 - 225 - 225 - 216 = 0$

and $h^2 - ab = (-6)^2 - 9.4 = 36 - 36 = 0$

Hence the given equation (1) represents a pair of straight lines.

Alternative Method:

$9x^2 - 12xy + 4y^2 - 15x + 10y + 6 = 0$ ... ...(1)

i.e., $(3x - 2y)^2 - 5(3x - 2y) + 6 = 0$

i.e., $u^2 - 5u + 6 = 0$ (where $u = 3x - 2y$)

i.e., $(u - 2)(u - 3) = 0$

i.e., $(3x - 2y - 2)(3x - 2y - 3) = 0$

Hence the given equation (1) represents a pair of lines whose individual equations are, $3x - 2y - 2 = 0$ and $3x - 2y - 3 = 0$

Which are clearly parallel because, $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ in this case.

Example 5: Show that the equation $x^2 - 2xy + y^2 - 4x + 4y + 3 = 0$ represents a pair of parallel straight lines. Find also the distance between them.

Solution: $x^2 - 2xy + y^2 - 4x + 4y + 3 = 0$ ... ...(1)

i.e., $(x - y)^2 - 4(x - y) + 3 = 0$

i.e., $u^2 - 4u + 3 = 0$ (where $u = x - y$)

i.e., $(u - 1)(u - 3) = 0$

i.e., $(x - y - 1)(x - y - 3) = 0$

Hence (1) represents a pair of straight lines where equations are,

$x - y - 1 = 0$ ... ...(2)

and $x - y - 3 = 0$ ... ...(3)

As the two equations differ only in their constant terms, they represent parallel straight line. So the given equation (1) represents a pair of parallel straight lines whose individual equations are (2) and (3).

Putting $y = 0$ in (2) we get $x = 1$; thus (1, 0) is a point on line (2).

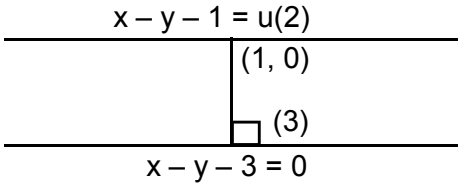


Fig. 3.1

Required distance between the parallel lines (2) and (3)

= Length of the perpendicular drawn from the point (1, 0) to the line (3)

$$= \left| \frac{1 - 0 - 3}{\sqrt{1^2 + (-1)^2}} \right| = \left| \frac{-2}{\sqrt{2}} \right| = \sqrt{2}$$



CHECK YOUR PROGRESS

Q.6: Prove that the equation $2x^2 - 7xy + 3y^2 + x + 7y - 6 = 0$, represents a pair of straight lines inclined to each other at 45^0.

Q.7: Prove that the equation $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$ represents a pair of parallel straight lines and find the distance between them.

Q.8: Find the value of k, so that the equation $6x^2 + xy - 12y^2 + 2x + ky - 20 = 0$ may represent a pair of straight lines.

Q.9: Find λ so that $\lambda x^2 + 3xy - 5y^2 + 7x + 14y + 3 = 0$ may represent a pair of straight lines.

Q.10: Prove that $2x^2 - 5xy + 3y^2 - 2x + 3y = 0$ represents a pair of lines. Find their point of intersection and the angle between them.



3.6 LET US SUM UP

- A homogeneous equation of 2nd degree represents a pair of straight lines through the origin.
- The separate equations of the two lines through the origin represented by $ax^2 + 2hxy + by^2 = 0$ are,

  $y - m_1x = 0$ and $y - m_2x = 0$

  i.e., $y = m_1x$ and $y = m_2x$

  Where $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.
- The angle θ between the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by $\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$

  i) The above two lines an coincident if $h^2 = ab$.

  ii) The two lines are perpendicular if $a + b = 0$.

- The equation of the pair of bisectors of the angles between the pair of lines $ax^2 + 2hxy + by^2 = 0$ is, $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$
- The condition for the general equation of second degree, viz. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is that $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
- If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, then the angle ϕ between then is given by $\tan\phi = \frac{2\sqrt{h^2 - ab}}{a + b}$
- $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents,
  a) a pair of intersecting lines if $\Delta = 0$ and $h^2 \neq ab$,
  b) a pair of parallel lines of $\Delta = 0$ and $h^2 = ab$,
  c) a pair of perpendicular lies if $\Delta = 0$ and $a + b = 0$,
  which $\Delta = abc + 2fgh - af^2 - by^2 - ch^2$.



3.7 FURTHER READING

1) Jain, P. K. and Ahmhad, K. *Textbook of Analytical Geometry of Three Dimensions*. New Delhi: New Age International (P) Ltd.
2) Kishan, Hari. *Coordinate Geometry of Two dimensions*. New Delhi: Atlantic Publishers & Distributors.
3) Pal, Sat and Pal, Harbans. *A Textbook of Two Dimensional Geometry*. New Age International Publishers.
4) Sharma, G. C. and Jain, Madhu. *Coordinate Geometry (2D and 3D)*. New Delhi: Galgotia Publications Pvt Ltd.
5) Sharma, A. K. *Coordinate Geometry (2D)*. New Delhi: Discovery Publishing House.



3.8 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: The required equation is $(3x - y)(x + 2y - 1) = 0$

i.e., $3x^2 + 5xy - 2y^2 - 3x + y = 0$

Ans. to Q. No. 2: i) The pair of lines is given by $8x^2 + 10xy + 3y^2 = 0$

Here a = 8, h = 5, b = 3.

Required angle between the pair of lines

$$\tan^{-1}\frac{2\sqrt{h^2-ab}}{a+b}$$

$$= \tan^{-1}\frac{2\sqrt{5^2-8.3}}{8+3}$$

$$= \tan^{-1}\frac{2}{11}$$

ii) The pair of lines is given by $x^2 - 2xy\sec\alpha + y^2 = 0$.

Here a = 1, h = $-\sec\alpha$, b = 1

Reqd. angle between the pair of lines

$$= \tan^{-1}\frac{2\sqrt{h^2-ab}}{a+b}$$

$$= \tan^{-1}\frac{2\sqrt{\sec^2\alpha-1}}{1+1}$$

$= \tan^{-1}(\tan\alpha) = \alpha$

Ans. to Q. No. 3: i) For the two lines $3x^2 - 8xy + \lambda y^2 = 0$ to be perpendicular to each other we must have.

$3 + \lambda = 0$ (cond^n: a + b = 0)

$\Rightarrow \lambda = -3$

ii) For the two lines $3x^2 - 8xy + \lambda y^2 = 0$ to be coincident we must have $(-4)^2 = 3.\lambda$ (cond^n: $h^2 = ab$)

$$\Rightarrow \lambda = \frac{16}{3}$$

Ans. to Q. No. 4: The equation of the bisectors of the angles between the pair of lines $ax^2 + 2hxy + by^2 = 0$ is, $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$

Hence,

i) The equation of the bisectors of the angles between the pair of lines $15x^2 - xy - 6y^2 = 0$ (for which a = 15, h = $-\frac{1}{2}$, b = –6) is,

$$\frac{x^2 - y^2}{15 - (-6)} = \frac{xy}{-\frac{1}{2}}$$

i.e., $\frac{x^2 - y^2}{21} = -2xy$

i.e., $x^2 - y^2 = -42xy$

i.e., $x^2 + 42xy - y^2 = 0$

ii) The equation of the bisectors of the angles between the pair of lines $x^2 - 3xy + 2y^2 = 0$ (for which a = 1, h = $-\frac{3}{2}$, b = 2) is,

$$\frac{x^2 - y^2}{1 - 2} = \frac{xy}{-\frac{3}{2}}$$

i.e., $x^2 - y^2 = \frac{2xy}{3}$

i.e., $3x^2 - 2xy - 3y^2 = 0$.

Ans. to Q. No. 5: It is given that the equation $x^2 + 4xy - y^2 = 0$ ... ...(i) is the equation of the bisectors of the angles between the pair of lines $x^2 + 2kxy - y^2 = 0$. But the equation of the bisectors of the angles between the above pair of lines is,

$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{k}$$

i.e., $kx^2 - ky^2 = 2xy$

i.e., $kx^2 - 2xy - ky^2 = 0$ ... ...(ii)

Hence the equations (i) & (ii) must be identical, so we get,

$$\frac{k}{1} = \frac{-2}{4} = \frac{-k}{-1}$$

$\Rightarrow k = -\frac{1}{2}$.

Ans. to Q. No. 6: $2x^2 - 7xy + 3y^2 + x + 7y - 6 = 0$ ... ...(i) comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, We have,

a = 2, h = $-\frac{7}{2}$, b = 3, g = $\frac{1}{2}$, f = $\frac{7}{2}$, c = –6.

Now, $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 2.3(-6) + 2.\left(\frac{7}{2}\right)^2 - 3.\left(\frac{1}{2}\right)^2 - (-6)\left(-\frac{7}{2}\right)^2$$

$$= -36 - \frac{49}{4} - \frac{49}{2} - \frac{3}{4} + \frac{147}{2}$$

$$= -36 - \frac{52}{4} + \frac{98}{2}$$

$$= -36 - 13 + 49$$

$$= 0$$

$\therefore$ The given equation (i) represents a pair of straight lines.

Also the angle between the above two lines

$$= \tan^{-1}\frac{2\sqrt{h^2 - ab}}{a + b}$$

$$= \tan^{-1}\frac{2\sqrt{\left(-\frac{7}{2}\right)^2 - 2.3}}{2 + 3}$$

$$= \tan^{-1}\frac{\sqrt{49 - 24}}{5}$$

$$= \tan^{-1}\frac{\sqrt{25}}{5}$$

$= \tan^{-1}1$

$= 45^0$

Ans. to Q. No. 7: $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$ ... ...(i)

i.e., $(x + 3y)^2 + 4(x+3y) - 5 = 0$

i.e., $u^2 + 4u - 5 = 0$ (where $u = x + 3y$)

i.e., $u^2 + 5u - u - 5 = 0$

i.e., $u(u + 5) - 1(u + 5) = 0$

i.e., $(u + 5)(u - 1) = 0$

i.e., $(x + 3y + 5)(x + 3y - 1) = 0$.

Hence (i) represents a pair of straight lines whose separate equations are

$x + 3y + 5 = 0$ ... ...(ii)

and $x + 3y - 1 = 0$ ... ...(iii)

As these two first degree equations differ only in their constant terms, they represent parallel straight lines. Hence the given equation (i) represents a pair of parallel straight lines whose individual equations are (ii) and (iii).

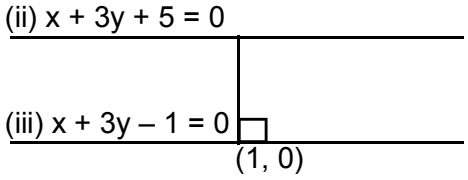


Fig. 3.2

Putting y = 0 in (iii) we get x = 1; thus (1, 0) is a point on the line (iii).

Reqd. distance between the parallel lines (ii) and (iii).

= Length of the perpendicular drawn from the point (1, 0) to the line (ii)

$$= \left|\frac{1+3.0+5}{\sqrt{1^2+3^2}}\right|$$

$$= \frac{6}{\sqrt{10}}$$

$$= \frac{3\sqrt{10}}{5}$$

Ans. to Q. No. 8: In order that the equation $6x^2 + xy - 12y^2 + 2x + ky - 20 = 0$ may represent a pair of straight lines we must have,

$$6.(-12).(-20) + 2.\frac{k}{2}.1.\frac{1}{2} - 6.\left(\frac{k}{2}\right)^2 - (-12).1^2 - (-20).\left(\frac{1}{2}\right)^2 = 0$$

$$\Rightarrow 1440 + \frac{k}{2} - \frac{3k^2}{2} + 12 + 5 = 0$$

$$\Rightarrow 2880 + k - 3k^2 + 34 = 0$$

$$\Rightarrow 3k^2 - k - 2914 = 0$$

$$\Rightarrow k = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.3.(-2914)}}{2 \times 3}$$

$$= \frac{1 \pm \sqrt{1 + 34968}}{6}$$

$$= \frac{1 \pm \sqrt{34969}}{6}$$

$= \frac{1 \pm 187}{6}$

$= \frac{188}{6}$ or, $\frac{-186}{6}$

$= \frac{94}{3}$ or, –31

$\therefore$ The required value of k is –31 or, $\frac{94}{3}$.

Ans. to Q. No. 9: If $\lambda x^2 + 3xy - 5y^2 + 7x + 14y + 3 = 0$ represents a pair of straight lines, then,

$$\lambda.(-5).3 + 2.7.\left(\frac{7}{2}\right).\left(\frac{3}{2}\right) - \lambda.7^2 - (-5).\left(\frac{7}{2}\right)^2 - 3.\left(\frac{3}{2}\right)^2 = 0$$

$$\Rightarrow -15\lambda + \frac{147}{2} - 49\lambda + \frac{245}{4} - \frac{27}{4} = 0$$

$$\Rightarrow -64\lambda + \frac{294 + 245 - 27}{4} = 0$$

$$\Rightarrow -64\lambda + \frac{512}{4} = 0$$

$\Rightarrow$ –64λ + 128 = 0

$\Rightarrow$ –64λ = –128

$\Rightarrow \lambda = 2$

$\therefore$ Reqd. value of λ is 2.

Ans. to Q. No. 10: $2x^2 - 5xy + 3y^2 - 2x + 3y = 0$ ... ...(i)

$\Rightarrow 2x^2 - (5y + 2)x + (3y^2 + 3y) = 0$

$$\Rightarrow x = \frac{(5y + 2) \pm \sqrt{(5y + 2)^2 - 8(3y^2 + 3y)}}{2.2}$$

$$= \frac{(5y + 2) \pm \sqrt{y^2 - 4y + 4}}{4}$$

$$= \frac{(5y + 2) \pm \sqrt{(y - 2)^2}}{4}$$

$$\Rightarrow x = \frac{(5y + 2) \pm (y - 2)}{4}$$

Corresponding to +sign, we get,

$$x = \frac{(5y+2)+(y-2)}{4} = \frac{6y}{4} = \frac{3y}{2}$$

i.e., $2x = 3y$

i.e., $2x - 3y = 0$;

Corresponding to –sign, we get,

$$x = \frac{(5y+2)-(y-2)}{4} = \frac{4y+4}{4}$$

i.e., $x = y + 1$

i.e., $x - y - 1 = 0$

Hence the given equation (i) represents a pair of straight lines whose individual equations are,

$2x - 3y = 0$ ... ...(ii)

and $x - y - 1 = 0$ ... ...(iii)

We have to solve (ii) and (iii) to find the coordinates of their point of intersection. From (ii) $y = \frac{2x}{3}$ and putting this in (iii), we get,

$$x - \frac{2x}{3} - 1 = 0$$

$\Rightarrow 3x - 2x - 3 = 0$

$\Rightarrow x = 3$

Hence, $y = \frac{2x}{3} = \frac{2\times 3}{3} = 2$

$\therefore$ The point of intersection of the pair of lines represected by (i) is (3, 2).

The line (ii), i.e., $y = \frac{2}{3}x$ has slope $m_1 = \frac{2}{3}$ and the line (iii), i.e., $y = x - 1$ has slope $m_2 = 1$; angle ϕ between them is given by

$$\tan\phi = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{2}{3} - 1}{1 + \frac{2}{3}.1}\right| = \left|\frac{-1/3}{5/3}\right| = \left|-\frac{1}{5}\right| = \frac{1}{5}$$

$$\Rightarrow \phi = \tan^{-1}\frac{1}{5}.$$



3.9 MODEL QUESTIONS

Q.1: Find the separate equations of the lines represented by the following equations:

i) $x^2 + 2xy\cos\alpha + y^2 = 0$ ii) $x^2 + 2xy\cot\theta + y^2 = 0$

Q.2: Find the equation to the lines through the origin and perpendicular to the lines $3x^2 - 8xy + 5y^2 = 0$.

Q.3: Find the equation to the lines through (2, 3) and perpendicular to the lines $3x^2 - 8xy + 5y^2 = 0$.

Q.4: Find the equation of the lines which bisects the angle between the axes.

Q.5: Find the straight lines given by the equation $x^3 + 3x^2y - 3xy^2 - y^3 = 0$.

Q.6: If the pair of straight lines $ax^2 + 2hxy - ay^2 = 0$ and $bx^2 + 2gxy - by^2 = 0$ be such that each bisects the angle between the other, then prove that $hg + ab = 0$.

Q.7: For what value of k does the equation $12x^2 + 2kxy + ky^2 + 11x - 5y + 2 = 0$ represent two straight lines.

Q.8: Show that the equation $6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0$ represents a pair of straight lines. Find their equations.

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UNIT 4: CIRCLE

UNIT STUCTURE

4.1 Learning Objective

4.2 Introduction

4.3 Standard Equation of a Circle

4.4 General Equation of a Circle

4.5 Equation of a Circle when End Points of its Diameter is Given

4.6 Tangent and Normal

4.7 Some Important Propositions

4.8 Let Us Sum Up

4.9 Further Reading

4.10 Answers to Check Your Progress

4.11 Model Questions

4.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- find standard equation of a circle
- find general equation of a circle
- find equation when end points of its diameter is given
- know about tangent and normal of a circle.

4.2 INTRODUCTION

A circle is the set of points which moves in a plane in such a way that its distance from a fixed point in that plane remains constant. In this unit, we will discuss equation of circle in various forms. We will also discuss equation of tangent and normal of a circle in this unit.

4.3 STANDARD EQUATION OF A CIRCLE

Let (x, y) be any point on the circle. Then its distance from the centre (h, k) is a, i.e., a is the radius.

$\therefore$ We have, $\sqrt{(x-h)^2+(y-k)^2}$ = a

Whence $(x-h)^2 + (y-k)^2 = a^2$ which is the equation of the circle.

As a particular case the equation of the circle whose centre is at the origin and whose radius is a is $x^2 + y^2 = a^2$ which is known as the standard equation of a circle.

For example the equation of the circle whose centre is (2, –3) and radius is, 7 is, $(x-2)^2 + (y+3)^2 = 7^2$

i.e., $x^2 - 4x + 4 + y^2 + 6y + 9 = 49$

i.e., $x^2 + y^2 - 4x + 6y - 36 = 0$

Note: When referred to the centre as the origin, the equation of a circle has the form $x^2 + y^2 = a^2$, known as the standard equation of the circle.

4.4 GENERAL EQUATION OF A CIRCLE

Consider the second degree equation

$x^2 + y^2 + 2gx + 2fx + c = 0$ ... ...(1)

Where, g, f and c are constants.

Now this equation can be expressed as,

$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$

i.e., $(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$

i.e., $\{x-(-g)\}^2 + \{y-(-1)\}^2 = \left(\sqrt{g^2+f^2-c}\right)^2$

Which represents a circle whose centre is the point (–g, –f) and radius $\sqrt{g^2+f^2-c}$.

(1) is the general equation of a circle. This circle is a real circle,a point circle or an imaginary circle according as, $g^2 + f^2 - c > 0, = 0, < 0$.

Hence, the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle whose centre is the point (–g, –f) and radius is $\sqrt{g^2+f^2-c}$.

Note: The most general equation of the 2nd degree in x and y is, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

This will represent a circle provided that (i) a = b and (ii) h = 0.

In other words (i) the coefficients of x^2 and y^2 must be equal and (ii) the equation should be free of the product term xy.

4.5 EQUATION OF A CIRCLE WHEN END POINTS OF ITS DIAMETER IS GIVEN

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of a circle. Let P(x, y) be any point on the circle. $\angle APB$, being an angle in a semicircle, is a right angle, i.e., AP is perpendicular to BP.

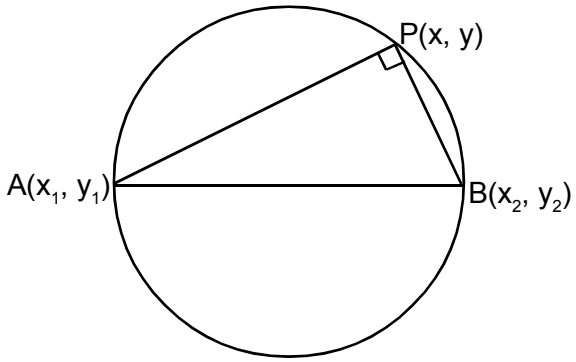


Fig. 4.1

Now the slope of the line PA is, $\frac{y - y_1}{x - x_1}$ and

that of the line PB is, $\frac{y - y_2}{x - x_2}$.

As AP $\perp$ BP, we get, $\left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$

i.e., $(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$

i.e., $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

This is the required equation of the circle.

ILLUSTRATIVE EXAMPLES:

Example 1: Examine in each of the following cases whether the given equation represents a circle. If so, find its centre and radius.

i) $2x^2 + 2y^2 - 4x + 6y + 1 = 0$

ii) $x^2 + y^2 - 4xy + 2x + 3y - 5 = 0$

Solution:

i) The equation $2x^2 + 2y^2 - 4x + 6y + 1 = 0$

i.e., $x^2 + y^2 - 2x + 3y + \frac{1}{2} = 0$

is of the form $x^2 + y^2 + 2gx + 2fx + c = 0$

where, 2g = –2, 2f = 3 and c = $\frac{1}{2}$.

Hence it represents a circle; its centre is (–g, –f), i.e., $\left(1,-\frac{1}{2}\right)$

and radius = $\sqrt{g^2+f^2-c}=\sqrt{1+\frac{9}{4}-\frac{1}{2}}=\sqrt{\frac{11}{4}}$.

ii) In the 2nd degree equation $x^2 + y^2 – 4xy + 2x + 3y – 5 = 0$

$h = –2 \neq 0$ (though a = b); hence it doesnot represent a circle.

Example 2:

i) Find the equation of the circle having a diameter whose end points are (0, –1) and (2, 3).

ii) Find the equation of the circle which is concentric with the above circle and whose radius is 5.

Solution:

i) End points of a diameter are (0, –1) and (2, 3). So the equation of the circle is, (x–0)(x–2) + (y+1)(y–3) = 0

i.e., $x^2 + y^2 – 2x – 2y – 3 = 0$

ii) The equation of a circle concentric with a given circle differs only in the constant term. Let the equation of the circle concentric with the given circle,

$x^2 + y^2 – 2x – 2y – 3 = 0$ be, $x^2 + y^2 – 2x – 2y – c = 0$;

Its radius = $\sqrt{(-1)^2+(-1)^2-c}=\sqrt{2-c}$.

For the required concentric circle, the radius is 5;

So, $\sqrt{2-c} = 5$

$\Rightarrow 2 – c = 25$

$\Rightarrow c = –23$

$\therefore$ The required equation of the circle is,

$x^2 + y^2 – 2x – 2y – 23 = 0$.

Example 3: Find the equation of the circle which passes through the origin and makes intercepts a and b on the axes of x and y respectively. State its centre and radius.

Solution:

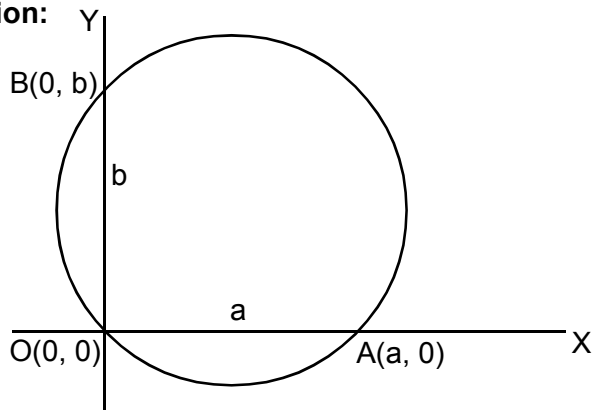


Fig. 4.2

The circle in question passes through the origin O and makes interceps OA = a on x-axis and OB = b on y-axis. Thus the circle passes through the three points O(0, 0), A(a, 0) and B(0, b). Let its equation be,

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... ...(1)

As it passes through O(0, 0), A(a, 0) and B(0, b) we must have,

$c = 0$ ... ...(2)

$a^2 + 2ga + c = 0$ ... ...(3)

and $b^2 + 2fb + c = 0$ ... ...(4)

Using (2) in (3) and (4) we get,

$2ga = -a^2 \Rightarrow g = -\frac{a}{2}$

and $2fb = -b^2 \Rightarrow f = -\frac{b}{2}$

Thus, $g = -\frac{a}{2}$, $f = -\frac{b}{2}$ and $c = 0$

Putting the values of g, f and c in (1), we get, $x^2 + y^2 - ax - by = 0$

Which is the required equation of the circle.

The centre of the circle is the point (–g, –f),

i.e., $\left(a/2, b/2\right)$ and its radius,

$= \sqrt{g^2 + f^2 - c} = \sqrt{\left(-\frac{a}{2}\right)^2 + \left(-\frac{b}{2}\right)^2 - 0}$

$= \frac{1}{2}\sqrt{a^2 + b^2}$.

Example 4: Write down the equation of the circle which has (2, –3) and (1, 0) as the end points of a diameter. What is its radius?

Solution: The equation of the circle is,

$(x – 2)(x – 1) + \{y – (–3)\}(y – 0) = 0$

i.e., $(x – 2)(x – 1) + y(y + 3) = 0$

i.e., $x^2 – 3x + 2 + y^2 + 3y = 0$

i.e., $x^2 + y^2 – 3x + 3y + 2 = 0$

Its radius $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - 2}$$

$$= \sqrt{\frac{9+9-8}{4}}$$

$$= \sqrt{\frac{10}{4}}$$

$$= \frac{1}{2}\sqrt{10}$$

4.6 TANGENT AND NORMAL

a) Consider the standard equation of a circle, viz.

$x^2 + y^2 = a^2$ ... ...(1)

and let $P(x_1, y_1)$ be a given point on it. Differentiating (1) w.r.t x, we get,

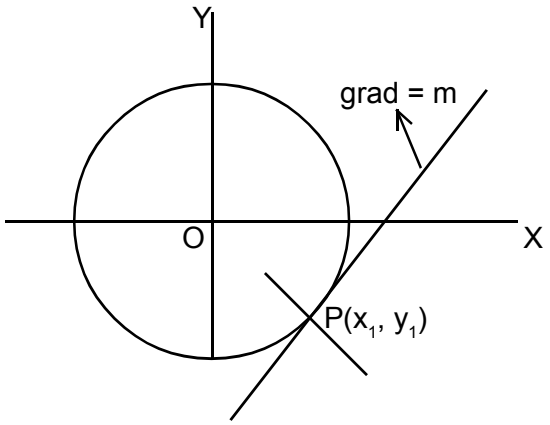


Fig. 4.3

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow\ 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow\ \frac{dy}{dx} = -\frac{x}{y}$$

From calculus we know that the gradient 'm' of the tangent to the circle (1) at the point $P(x_1, y_1)$ on it is given by

$$m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = -\frac{x_1}{y_1}$$

The tangent at P is a line having gradient 'm' and passing through the point $P(x_1, y_1)$, and hence its equation is,

$$y - y_1 = m(x - x_1)$$

i.e., $y - y_1 = \left(-\frac{x_1}{y_1}\right)(x - x_1)$

i.e., $yy_1 - y_1^2 = -xx_1 + x_1^2$

i.e., $xx_1 + yy_1 = x_1^2 + y_1^2$

But $P(x_1, y_1)$ is a point on the circle (1) and so we have $x_1^2 + y_1^2 = a^2$; hence the equation of the tangent at $P(x_1, y_1)$ is $xx_1 + yy_1 = a^2$

The normal to the circle at $P(x_1, y_1)$ is the line through $P(x_1, y_1)$ at right angles to the tangent at P; so the gradient of the normal at P is $-\frac{1}{m}$.

Hence the equation of the normal to the circle (1) at $P(x_1, y_1)$ is,

$$y - y_1 = \left(-\frac{1}{m}\right)(x - x_1)$$

i.e., $y - y_1 = \left(-\frac{1}{-\frac{x_1}{y_1}}\right)(x - x_1) = \frac{y_1}{x_1}(x - x_1)$

i.e., $\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1}$

i.e., $\frac{x}{x_1} = \frac{y}{y_1}$

Remarks: The equation $\frac{x}{x_1} = \frac{y}{y_1}$ is clearly satisfied by x = 0, y = 0.

Thus the normal to the circle $x^2 + y^2 = a^2$ at (x_1, y_1) passes through the origin, which is the centre of the circle. As the equation of any circle can be put in the form $x^2 + y^2 = a^2$ choosing rectangular coordinate axes through its centre as the origin, it follows that any normal to a circle passes through its centre.

b) Next consider the general equation of a circle:

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... ...(2)

Differentiating both sides of (2) w.r. to x we get

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\Rightarrow 2(y + f)\frac{dy}{dx} = -2(x + g)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x + g}{y + f}.$$

If (x_1, y_1) is a given point on this circle and if m be the gradient of the tangent to it at (x_1, y_1), then,

$$m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = -\frac{x_1 + g}{y_1 + f},$$

The equation of the tangent at (x_1, y_1) is,

$y - y_1 = m(x - x_1)$

i.e., $y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$

i.e., $(x - x_1)(x_1 + g) = -(y - y_1)(y_1 + f)$

i.e., $(xx_1 + gx - x_1^2 - gx_1) + (yy_1 + fy - y_1^2 - fx_1) = 0$

i.e., $xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$ ... ...(3)

But (x_1, y_1) is a point on the circle (2) so that we have,

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

Whence, $x_1^2 + y_1^2 + gx_1 + fy_1 = -gx_1 - fy_1 - c$.

Substituting this in the R.H.S. of (3) we get,

$xx_1 + yy_1 + gx + fy = -gx_1 - fy_1 - c$

i.e., $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

This is the equation of the tangent to the circle (2) at a given point (x_1, y_1) on it.

The gradient m' of the normal to the circle (2) at (x_1, y_1) is given by

$m' = -\frac{1}{m} = \frac{y_1 + f}{x_1 + g}$ ($\because$ normal is $\perp$ to tangent),

So, the equation of the normal at (x_1, y_1) is,

$y - y_1 = m'(x - x_1)$

i.e., $y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$

i.e., $\frac{x - x_1}{g + x_1} = \frac{y - y_1}{f + y_1}$

Example 1: Write down the equations of the tangent and normal to the circle $x^2 + y^2 = 5$ at the point (1, –2) on it.

Solution: The equation of the tangent to the circle at (1, –2) is,

$x.1 + y.(-2) = 5$

i.e., $x - 2y = 5$

and that of the normal at (1, –2) is, $\frac{x}{1} = \frac{y}{-2}$

i.e., $2x + y = 0$.

Example 2: Show that (3, 2) is a point on the circle $x^2 + y^2 - x + 4y - 18 = 0$. Find the equation of its tangent at (3, 2). Verify also that the normal at (3, 2) passes through the centre of the circle.

Solution: When $x = 3$ and $y = 2$,

$x^2 + y^2 - x + 4y - 18 = 9 + 4 - 3 + 8 - 18 = 0$;

So (3, 2) is a point on the circle $x^2 + y^2 - x + 4y - 18 = 0$ ... ...(1)

Here, $2g = -1$, $2f = 4$, $c = -18$

$\Rightarrow g = -\frac{1}{2}$, $f = 2$, $c = -18$

Also, $(x_1, y_1) \equiv (3, 2)$, i.e., $x_1 = 3$, $y_1 = 2$

The equation of the tangent to the given circle at (3, 2) is,

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

i.e., $x.3 + y.2 + \left(-\frac{1}{2}\right)(x + 3) + 2(y + 2) - 18 = 0$

i.e., $6x + 4y - x - 3 + 4y + 8 - 36 = 0$

i.e., $5x + 8y - 31 = 0$

The equation of the normal to the given circle at (3, 2) is,

$$\frac{x - x_1}{g + x_1} = \frac{y - y_1}{f + y_1}$$

i.e., $$\frac{x - 3}{-\frac{1}{2} + 3} = \frac{y - 2}{2 + 2}$$

i.e., $$\frac{2(x - 3)}{5} = \frac{y - 2}{4}$$

i.e, $8x - 24 = 5y - 10$

i.e., $8x - 5y - 14 = 0$

The centre of the circle is (–g, –f) i.e., $\left(\frac{1}{2}, -2\right)$ and its coordinates clearly satisfy the above equation. Hence it is verified that the normal at (3, 2) passes through the centre of the circle.



CHECK YOUR PROGRESS

Q.1: Find the equation of the circle passing through the points (4, 3) and (–2, 5), and having the line $2x - 3y = 4$ as a diameter.

Q.2: Find k so that the equation $16x^2 + 25y^2 - 1 = k(x - 2)^2$ may represent a circle.

Q.3: The end points of a diameter of a circle are (5, 1) and (7, 5). Find the equation of the circle concentric with the given circle and having radius 10.

Q.4: Find the equation of the circle which passes through the points (–1, 1), (–2, 1), (4, 3).

Q.5: Prove that the line $2x + y = 4$ is a tangent to the circle $x^2 + y^2 + 6x - 10y + 29 = 0$. Also find the point of contact.

4.7 SOME IMPORTANT PROPOSITIONS

a) To find the condition that the line lx + my + n = 0 may be a tangent to the circle $x^2 + y^2 = a^2$

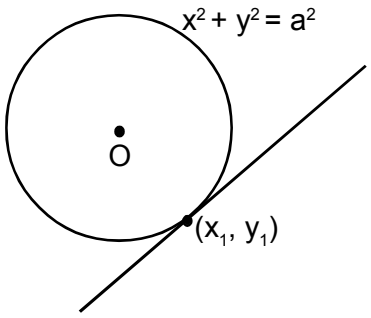


Fig. 4.4

Suppose that the line, lx + my + n = 0 ... ...(i)

is a tangent to the circle $x^2 + y^2 = a^2$, (x_1, y_1) being the point of contact.

Then, (i) must be identical with the equation of the tangent at (x_1,y_1), viz.

$xx_1 + yy_1 = a^2$

i.e., $xx_1 + yy_1 - a^2 = 0$ ... ...(ii)

Comparing the equations (i) and (ii), which are identical, we get,

$$\frac{x_1}{l} = \frac{y_1}{m} = \frac{-a^2}{n}$$

$$\Rightarrow x_1 = -\frac{a^2 l}{n}, y_1 = -\frac{a^2 m}{n}$$

As (x_1, y_1) is a point on the circle $x^2 + y^2 = a^2$, we get,

$x_1^2 + y_1^2 = a^2$

$$\Rightarrow \left(-\frac{a^2 l}{n}\right)^2 + \left(-\frac{a^2 m}{n}\right) = a^2$$

$\Rightarrow a^2(l^2 + m^2) = n^2$,

Which is the required condition of tangency.

Note:

i) When the above condition is satisfied, the line lx + my + n = 0 touches the circle $x^2 + y^2 = a^2$, and in that event the point of contact is,

$$\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$$

ii) The condition for the line $y = mx + c$, i.e., $mx – y + c = 0$ to be a tangent to the circle $x^2 + y^2 = a^2$ is, $a^2 \{m^2 + (–1)^2\} = c^2$

i.e., $c^2 = a^2(1 + m^2)$.

Accordingly the pair of parallel lines $y = mx \pm a\sqrt{1+m^2}$ always touch the circle $x^2 + y^2 = a^2$; m is the slope of either tangent.

b) To prove that exactly two tangents can be drawn from a given point to a given circle.

Let $x^2 + y^2 = a^2$ be the given circle and (x_1, y_1) be a given point.

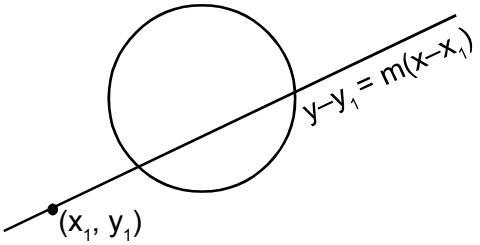


Fig. 4.5

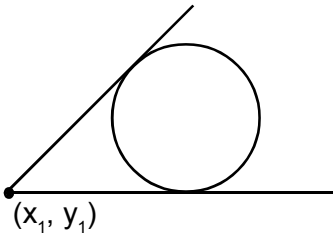


Fig. 4.6

Any line through (x_1, y_1) is $y – y_1 = m(x – x_1)$

i.e., $y = mx + (y_1 – mx_1)$

and this will be a tangent to the circle $x^2 + y^2 = a^2$ if

$(y_1–mx_1)^2 = a^2(1+m^2)$ [Condition $c^2 = a^2(1+m^2)$]

i.e., $(x_1^2 – a^2)m^2 – 2x_1y_1m + (y_1^2 - a^2) = 0$.

This being a quadratic equation in m gives exactly two values of m, each of which gives us a tangent to the circle. Hence exactly two tangents can be drawn from the given point (x_1, y_1) to the given circle.

Note:

i) The two tangents are real and distinct, real and coincident, or imaginary according as the roots of the above quadatic in m an real & distinct, real & equal, or imaginary, i.e., according as $(–2x_1 y_1)^2 – 4(x_1^2 – a^2)(y_1^2 – a^2) \gtreqless 0$, i.e. according as $x_1^2 + y_1^2 \gtreqless a^2$,

i.e., according as the point (x_1, y_1) lies outside, on or inside the circle.

ii) It can be established that the equation of the pair of tangents drawn from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $SS_1 = T^2$

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$, $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ and $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

c) Chord of contact:

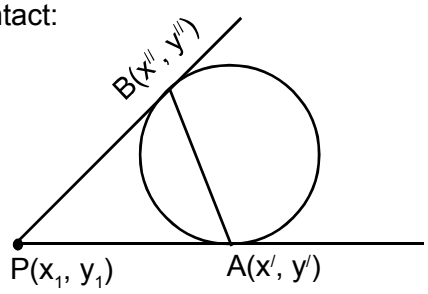


Fig. 4.7

Let P be a given point outside a given circle so that two real tangents can be drawn from P to the circle. If these two tangents touch the circle at A and B, then the line segment AB joining the points of contact is called the chord of contact of the two tangents drawn from the point P w.r. to the given circle.

Let the circle be $x^2 + y^2 = a^2$ and (x_1, y_1) be the coordinates of the given point P outside it. Let the points of contact of the two tangents that can be drawn for $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ be $A(x', y')$ and $B(x'', y'')$.

The tangent at $A(x', y')$ is $xx' + yy' = a^2$

and the tangent at $B(x'', y'')$ is $xx'' + yy'' = a^2$.

As these tangents pass through $P(x_1, y_1)$, we get $x'x_1 + y'y_1 = a^2$ and $x'' x_1 + y''y_1 = a^2$, whence it is clear that the coordinates of both A and B satisfy the equation $xx_1 + yy_1 = a^2$, which represents a line. Hence this is the line through A and B.

Therefore the equation of the chord of contact AB of the two tangents drawn from $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

Note: It can be proved that the equation of the chord of contact of the two tangents drawn from the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is, $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

d) Common chord of two circles:

Consider the two circles

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$.

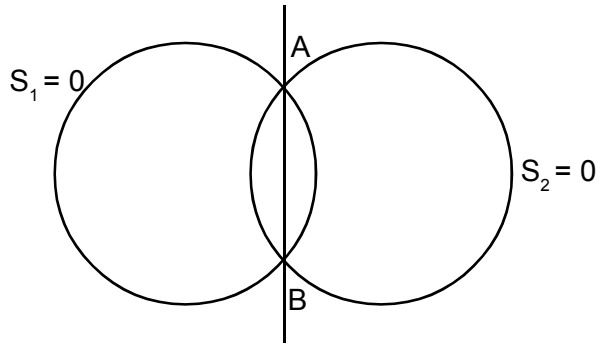


Fig. 4.8

In the above figure AB is their common chord.

Consider the equation: $S_1 - S_2 = 0$

i.e., $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$, which, being a first degree equation in x and y, represents a straight line. Now the coordinates of A satisfy both $S_1 = 0$ and $S_2 = 0$ and hence satisfy the equation $S_1 - S_2 = 0$. So, A lies on the line $S_1 - S_2 = 0$. Similarly B lies on the line $S_1 - S_2 = 0$. It follows that the line $S_1 - S_2 = 0$ passes through both A and B; in other words the equation of the line AB, which is the common chord of the circles $S_1 = 0$ and $S_2 = 0$, is $S_1 - S_2 = 0$.

Note that in each of the equation $S_1 = 0$ and $S_2 = 0$, the coefficients of x^2 and y^2 are unity.

The common chord of two circles is also known as their radical axis.

e) Condition for two circles to touch each other:

Consider the two circles:

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

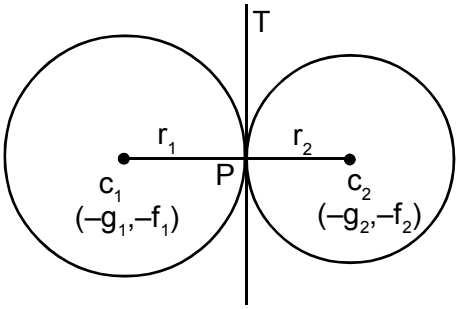


Fig. 4.9

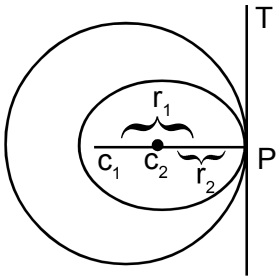


Fig. 4.10

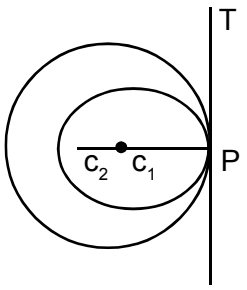


Fig. 4.11

$C_1(-g_1, -f_1)$ is the centre of the first circle and its radius is $r_1 = \sqrt{g_1^2 + f_1^2 - c_1}$ while $C_2(-g_2, -f_2)$ is the centre of the second circle

and its radius is $r_2 = \sqrt{g_2^2 + f_2^2 - c_2}$. When these two circles touch each other at P, they have a common tangent PT at P to which both C_1P and C_2P are perpendicular and hence C_1, P, C_2 lie in the same straight line. In case they touch each other externally at P, We have,

$C_1C_2 = C_1P + C_2P = r_1 + r_2$

i.e., the distance between the centres is equal to the sum of the radii of the two circles; if they touch each other internally, we have,

$C_1C_2 = C_1P - C_2P$ or, $C_1C_2 = C_2P - C_1P$

$= r_1 - r_2$ $= r_2 - r_1$

i.e., $C_1C_2 = r_1 \sim r_2$

i.e., the distance between the centres of the two circles is equal to absolute difference of their radii. These two conditions can be combined as, $C_1C_2^2 = (r_1 \pm r_2)^2$

Which can be analytically expressed as,

$$(g_2 - g_1)^2 + (f_2 - f_1)^2 = \left(\sqrt{g_1^2 + f_1^2 - c_1} \pm \sqrt{g_2^2 + f_2^2 - c_2}\right)^2$$

Example 1: Find the equations of the two tangents to the circle $x^2 + y^2 = 4$ which make an angle of 60^0 with the x-axis.

Solution: We know that the equations of the two tangents of the circle $x^2 + y^2 = a^2$ with a given slope m are $y = mx \pm a\sqrt{1+m^2}$.

Here the given circle is $x^2 + y^2 = 4$. So that a = 2 and the slope of either tangent is $m = \tan 60^0 = \sqrt{3}$. Hence the required equations of the two tangents are, $y = \sqrt{3}x \pm 2\sqrt{1+3}$, i.e., $y = \sqrt{3}x \pm 4$.

Example 2: Show that the two circles $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch each other externally.

Solution: The centre of the circle $x^2 + y^2 - 4x + 6y + 8 = 0$ is $C_1(2, -3)$ and its radius is, $r_1 = \sqrt{(-2)^2 + 3^2 - 8} = \sqrt{5}$.

The centre of the circle $x^2 + y^2 - 10x - 6y + 14 = 0$ is $C_2(5, 3)$ and its radius is, $r_2 = \sqrt{(-5)^2 + (-3)^2 - 14} = \sqrt{20} = 2\sqrt{5}$.

The distance between the centres:

= $C_1C_2 = \sqrt{(5-2)^2+(3+3)^2} = \sqrt{9+36}$

= $\sqrt{45} = 3\sqrt{5} = \sqrt{5}+2\sqrt{5}$

= $r_1 + r_2$

= the sum of the radii

$\therefore$ The two circles touch each other externally.



CHECK YOUR PROGRESS

Q.6: For what value of λ does the line $4x + \lambda y + 7 = 0$ touch the circle $x^2 + y^2 - 6x + 4y - 12 = 0$?

Q.7: If the two circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch, show that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$.

Q.8: Find the equation of the circle whose diameter is the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.

Q.9: Prove that the pair of tangents from the point (1, 2) to the circle $x^2 + y^2 - 4x + 2y = 0$ are mutually perpendicular.

Q.10: Find the equation of the chord of contact of the point (4, 3) w.r. to the circle $x^2 + y^2 - 3x + 12y + 6 = 0$.



4.8 LET US SUM UP

- If (x_1, y_1) and (x_2, y_2) are the end points of a diameter of a circle, then the equation of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.
- The equation of the tangent to the circle $x^2 + y^2 = a^2$ at a given point (x_1, y_1) on it is $xx_1 + yy_1 = a^2$ and that of the normal at (x_1, y_1) is, $\frac{x}{x_1} = \frac{y}{y_1}$.
- If (x_1, y_1) is a given point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of its tangent at (x_1, y_1) is, $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ and the equation of its normal at (x_1, y_1) is, $\frac{x - x_1}{g + x_1} = \frac{y - y_1}{f + y_1}$.



4.9 FURTHER READING

1) Jain, P. K. and Ahmhad, K. *Textbook of Analytical Geometry of Three Dimensions*. New Delhi: New Age International (P) Ltd.
2) Kishan, Hari. *Coordinate Geometry of Two dimensions*. New Delhi: Atlantic Publishers & Distributors.
3) Pal, Sat and Pal, Harbans. *A Textbook of Two Dimensional Geometry*. New Age International Publishers.
4) Sharma, G. C. and Jain, Madhu. *Coordinate Geometry (2D and 3D)*. New Delhi: Galgotia Publications Pvt Ltd.
5) Sharma, A. K. *Coordinate Geometry (2D)*. New Delhi: Discovery Publishing House.



4.10 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: Let the required equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... ...(1)

As it passes through the points (4, 3) and (–2, 5), we have

$4^2 + 3^2 + 2g.4 + 2f.3 + c = 0$ and $(–2)^2 + 5^2 + 2g.(–2) + 2f.5 + c = 0$

i.e., $8g + 6f + c + 25 = 0$ ... ...(2)

and $– 4g + 10f + c + 29 = 0$ ... ...(3)

Further the line $2x – 3y = 4$ is a diameter of the circle, so the above line passes through the centre $(–g, –f)$ of the circle (1) giving us,

$2(–g) – 3(–f) = 4$

i.e., $–2g + 3f – 4 = 0$ ... ...(4)

We shall solve (2), (3) and (4) to get the values of the constants g, f, c, Subtracting (3) from (2), we get,

$12g – 4f – 4 = 0$

$\Rightarrow 3g – f – 1 = 0$ ... ...(5)

Multiplying (5) by 3,

$9g – 3f – 3 = 0$ ... ...(6)

Adding (4) and (6), $7g - 7 = 0 \Rightarrow 7g = 7 \Rightarrow g = 1$.

Thus from (5), $3.1 - f - 1 = 0 \Rightarrow f = 2$.

Hence (2) gives, $8.1 + 6.2 + c + 25 = 0 \Rightarrow c = -45$.

Thus, $g = 1, f = 2, c = -45$.

putting these values of g, f, c in (1). We get,

$x^2 + y^2 + 2x + 4y - 45 = 0$ as the required equation of the circle.

Alternative Method: Let (h, k) be the centre of the circle and a be its radius. As the circle passes through the points (4, 3) and (–2, 5).

We have, $(h - 4)^2 + (k - 3)^2 = a^2$

and $(h + 2)^2 + (k - 5)^2 = a^2$

i.e., $h^2 + k^2 - 8h - 6k + 25 = a^2$ ... ...(1)

and $h^2 + k^2 + 4h - 10k + 29 = a^2$ ... ...(2).

Subtracting (1) from (2),

$12h - 4k + 4 = 0$

$\Rightarrow 3h - k + 1 = 0$ ... ...(3)

Again as the line $2x - 3y = 4$ is a diameter of the circle, the line passes through the centre (h, k) of the circle. So,

$2h - 3k - 4 = 0$ ... ...(4)

From (3) and (4), viz. $\left.\begin{matrix} 3h - k + 1 = 0 \\ 2h - 3k - 4 = 0 \end{matrix}\right\}$

We get, by the method of cross multiplication,

$$\frac{h}{4+3} = \frac{k}{2+12} = \frac{1}{-9+2}$$

$$\Rightarrow \frac{h}{7} = \frac{k}{14} = \frac{1}{-7}$$

$\Rightarrow h = -1, \; k = -2$.

Hence, $a^2 = (h - 4)^2 + (k - 3)^2$

$= (-5)^2 + (-5)^2 = 50$

$\Rightarrow a = \sqrt{50}$

Thus the centre of the circle is the point (h, k), i.e., (–1, –2) and its radius is $a = \sqrt{50}$. So the required equation of the circle is,

$$(x + 1)^2 + (y + 2)^2 = \left(\sqrt{50}\right)^2$$

i.e., $x^2 + y^2 + 2x + 4y - 45 = 0$

Ans. to Q. No. 2: The given equation is,

$16x^2 + 25y^2 - 1 = k(x - 2)^2$

i.e., $(16 - k)x^2 + 25y^2 + 4kx - (4k + 1) = 0$.

This is a 2nd degree equation in x and y without xy term. It will represent a circle provided the coefficients of x^2 and y^2 are equal, i.e., provided $16 - k = 25$, i.e., $k = -9$.

$\therefore$ Required value of k is –9

Ans. to Q. No 3: The given circle has equation

$(x - 5)(x - 7) + (y - 1)(y - 5) = 0$

i.e., $(x^2 - 12x + 35) + (y^2 - 6y + 5) = 0$

i.e., $x^2 + y^2 - 12x - 6y + 40 = 0$.

The required circle is concentric with the given circle and so its equation is of the form $x^2 + y^2 - 12x - 6y + c = 0$.

As its radius is 10, we must have,

$$\sqrt{(-6)^2 + (-3)^2 - c} = 10$$

$\Rightarrow \sqrt{45 - c} = 10$

$\Rightarrow 45 - c = 100$

$\Rightarrow c = -55$

So the required equation of the circle is, $x^2 + y^2 - 12x - 6y - 55 = 0$.

Ans. to Q. No. 4: Let the required equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... ...(1)

As the circle passes through (–1, 1), (–2, 1) and (4, 3), we get,

$$\left.\begin{aligned} 1 + 1 - 2g + 2f + c = 0 &\Rightarrow -2g + 2f + c = -2 \\ 4 + 1 - 4g + 2f + c = 0 &\Rightarrow -4g + 2f + c = -5 \\ 16 + 9 - 8g + 6f + c = 0 &\Rightarrow -8g + 6f + c = -25 \end{aligned}\right\}$$

By Cramer's rule we get, $g = \frac{D_1}{D}$, $f = \frac{D_2}{D}$, $c = \frac{D_3}{D}$

Where, $D = \begin{vmatrix} -2 & 2 & 1 \\ -4 & 2 & 1 \\ 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ -12 & -4 & 0 \\ 8 & 6 & 1 \end{vmatrix} \left(\begin{matrix} R_1' = R_1 - R_2, \\ R_2' = R_2 - R_3 \end{matrix}\right)$

$= -8$,

$$D_1 = \begin{vmatrix} -2 & 2 & 1 \\ -5 & 2 & 1 \\ -25 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 20 & -4 & 0 \\ -25 & 6 & 1 \end{vmatrix} = -12$$

$$D_2 = \begin{vmatrix} -2 & -2 & 1 \\ -4 & -5 & 1 \\ 8 & -25 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 0 \\ -12 & 20 & 0 \\ 8 & -25 & 1 \end{vmatrix} = 40 + 36 = 76,$$

$$D_3 = \begin{vmatrix} -2 & 2 & -2 \\ -4 & 2 & -5 \\ 8 & 6 & -25 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & -2 \\ -2 & -3 & -5 \\ 14 & -19 & -25 \end{vmatrix} (C_1' = C_1 + C_2, C_2' = C_2 + C_3)$$

$= -2(3 + 42) = -160.$

$$\therefore g = \frac{D_1}{D} = \frac{-12}{-8} = \frac{3}{2},$$

$$f = \frac{D_2}{D} = \frac{76}{-8} = -\frac{19}{2},$$

$$c = \frac{D_3}{D} = \frac{-160}{-8} = 20.$$

$\therefore$ (1) becomes,

$$x^2 + y^2 + 2.\frac{3}{2}x + 2.\left(-\frac{19}{2}\right)y + 20 = 0$$

i.e., $x^2 + y^2 + 3x - 19y + 20 = 0$

This is the required equation of the circle.

Ans. to Q. No. 5: The given line is, $2x + y = 4$

i.e., $2x + y - 4 = 0$ ... ...(1)

and the given circle is, $x^2 + y^2 + 6x - 10y + 29 = 0$ ... ...(2)

whose centre is C(–g, –f), i.e., C(–3, 5)

and radius = $\sqrt{g^2 + f^2 - c}$

$= \sqrt{3^2 + (-5)^2 - 29}$

$= \sqrt{5}$

The length of the perpendicular drawn from the centre C(–3, 5) of the circle (2) to the line (1),

$$= \left|\frac{2.(-3)+5-4}{\sqrt{2^2+1^2}}\right| = \left|\frac{-5}{\sqrt{5}}\right|$$

$= \sqrt{5}$ = the radius of the circle.

$\therefore$ The line (1) is a tangent to the circle (2).

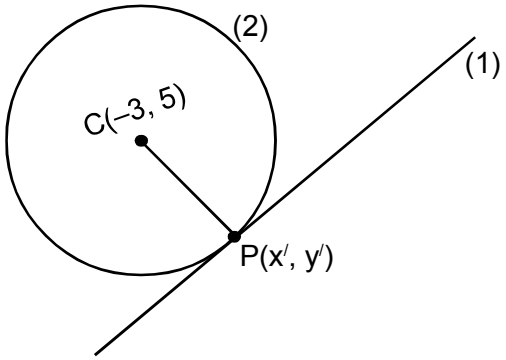


Fig. 4.12

Let P(x′, y′) be the point of contact. Then the line CP is perpendicular to the tangent line (1).

But, m = gradient of the line $CP = \frac{y' - 5}{x' + 3}$

and m' = gradient of the line (1) = –2.

As the above two lines are perpendicular we have,

$mm' = -1$

$$\Rightarrow \left(\frac{y' - 5}{x' + 3}\right)(-2) = -1$$

$\Rightarrow x' + 3 = 2y' - 10$

$\Rightarrow x' - 2y' + 13 = 0$

Again the point P(x′, y′) lies on the line (1) so that, we have,

$2x' + y' - 4 = 0$

Solving the equations,

and $\left.\begin{aligned} x' - 2y' + 13 &= 0 \\ 2x' + y' - 4 &= 0 \end{aligned}\right\}$

We get, $\frac{x'}{8-13} = \frac{y'}{26+4} = \frac{1}{1+4}$

$$\Rightarrow \quad \frac{x'}{-5} = \frac{y'}{30} = \frac{1}{5}$$

$$\Rightarrow \quad x' = -1, \; y' = 6$$

$\therefore$ The point of contact is (–1, 6).

Ans. to Q. No. 6: The circle is, $x^2 + y^2 - 6x + 4y - 12 = 0$; its centre is the point (3, –2) and radius = $\sqrt{9+4-(-12)} = \sqrt{25} = 5$.

In order that the line $4x + \lambda y + 7 = 0$ may touch the above circle, the length of the perpendicular drawn from the centre (3, –2) of the circle to the above line must be equal to radius of the circle, thus we must have.

$$\left|\frac{4.3 + \lambda.(-2) + 7}{\sqrt{4^2 + \lambda^2}}\right| = 5$$

$$\Rightarrow \quad \left|\frac{19 - 2\lambda}{\sqrt{16 + \lambda^2}}\right| = 5$$

$$\Rightarrow \quad \frac{19 - 2\lambda}{\sqrt{16 + \lambda^2}} = \pm 5$$

$$\Rightarrow \quad \left(\frac{19 - 2\lambda}{\sqrt{16 + \lambda^2}}\right)^2 = 25$$

$$\Rightarrow \quad 361 - 76\lambda + 4\lambda^2 = 400 + 25\lambda^2$$

$$\Rightarrow \quad 21\lambda^2 + 76\lambda + 39 = 0$$

$$\Rightarrow \quad \lambda = \frac{-76 \pm \sqrt{(76)^2 - 4.21.39}}{2.21} = \frac{-76 \pm \sqrt{5776 - 3276}}{42}$$

$$= \frac{-76 \pm \sqrt{2500}}{42} = \frac{-76 \pm 50}{42}$$

$$= \frac{-126}{42} \text{ or, } \frac{-26}{42} = -3 \text{ or, } -\frac{13}{21}$$

$\therefore$ The required value of λ is – 3 or, $-\frac{13}{21}$

Ans. to Q. No. 7: The centre of the circle $x^2 + y^2 + 2ax + c = 0$ is (–a, 0) and its radius is $\sqrt{a^2 - c}$; the centre of the circle $x^2 + y^2 + 2by + c = 0$ is (0, –b) and its radius is $\sqrt{b^2 - c}$. As the two circles touch each other,

the distance between their centres is equal to the sum or difference of their radii. Thus we get,

$$\sqrt{a^2+b^2} = \sqrt{a^2-c^2} \pm \sqrt{b^2-c^2}$$

$$\Rightarrow \left(\sqrt{a^2+b^2}\right)^2 = \left(\sqrt{a^2-c} \pm \sqrt{b^2-c}\right)^2$$

[Note that $(p \sim q)^2 = |p-q|^2 = (p-q)^2$]

$$\Rightarrow a^2+b^2 = (a^2-c)+(b^2-c) \pm 2\sqrt{(a^2-c)(b^2-c)}$$

$$\Rightarrow \pm 2\sqrt{(a^2-c)(b^2-c)} = 2c$$

$$\Rightarrow \pm \sqrt{(a^2-c)(b^2-c)} = c$$

$\Rightarrow (a^2 - c)(b^2 - c) = c^2$ (on squaring)

$\Rightarrow a^2b^2 - a^2c - b^2c + c^2 = c^2$

$\Rightarrow c(a^2 + b^2) = a^2b^2$

$$\Rightarrow \frac{b^2+a^2}{a^2b^2} = \frac{1}{c}$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$$

Ans. to Q. No. 8: The given circles are,

$S \equiv x^2 + y^2 + 2x + 3y + 1 = 0$ ... ...(1)

and $S' \equiv x^2 + y^2 + 4x + 3y + 2 = 0$ ... ...(2)

The equation of their common chord is, $S - S' = 0$

i.e., $-2x - 1 = 0$, i.e., $2x + 1 = 0$ ... ...(3)

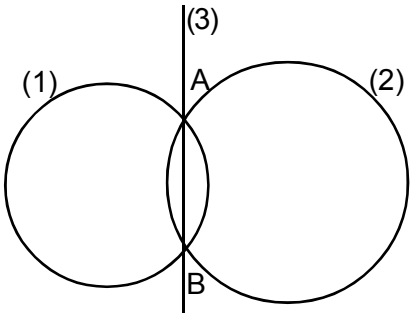


Fig. 4.13

In the above diagram AB is the common chord represented by the equation (3). Solving (1) and (3) we get the coordinates of A and B.

(3) $\Rightarrow x = -\frac{1}{2}$, putting this in (1)

We get, $\left(-\frac{1}{2}\right)^2 + y^2 + 2.\left(-\frac{1}{2}\right) + 3y + 1 = 0$

$\Rightarrow\ y^2 + 3y + \frac{1}{4} = 0$

$\Rightarrow\ 4y^2 + 12y + 1 = 0$

$\Rightarrow\ y = \frac{-12 \pm \sqrt{144 - 16}}{2.4}$

$= \frac{-12 \pm \sqrt{128}}{8}$

$= \frac{-12 \pm 8\sqrt{2}}{8}$

$= -\frac{3}{2} \pm \sqrt{2}$

$\therefore \left(-\frac{1}{2}, -\frac{3}{2} + \sqrt{2}\right)$ and $\left(-\frac{1}{2}, -\frac{3}{2} - \sqrt{2}\right)$ are the coordinates of A and B.

$\therefore$ The equation of the circle whose diameter is the common chord AB is,

$$\left\{x - \left(-\frac{1}{2}\right)\right\}\left\{x - \left(-\frac{1}{2}\right)\right\} + \left\{y - \left(-\frac{3}{2} + \sqrt{2}\right)\right\}\left\{y - \left(-\frac{3}{2} - \sqrt{2}\right)\right\} = 0$$

i.e., $\left(x + \frac{1}{2}\right)^2 + \left\{\left(y + \frac{3}{2}\right) - \sqrt{2}\right\}\left\{\left(y + \frac{3}{2}\right) + \sqrt{2}\right\} = 0$

i.e., $\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 - \left(\sqrt{2}\right)^2 = 0$

i.e., $x^2 + x + \frac{1}{4} + y^2 + 3y + \frac{9}{4} - 2 = 0$

i.e., $x^2 + y^2 + x + 3y + \frac{1}{2} = 0$

i.e., $2x^2 + 2y^2 + 2x + 6y + 1 = 0$

Ans. to Q. No. 9: The equation of the pair of tangents from the point (1, 2) to the circle, $x^2 + y^2 - 4x + 2y = 0$ is $SS_1 = T^2$

Where, $S \equiv x^2 + y^2 - 4x + 2y$,

$S_1 \equiv S(1, 2) = 1^2 + 2^2 - 4.1 + 2.2 = 5$

$T \equiv x.1 + y.2 - 2(x+1) + 1(y+2)$

$= x + 2y - 2x - 2 + y + 2$

$= 3y - x.$

So the pair of tangents from (1, 2) is given by

$(x^2 + y^2 - 4x + 2y).5 = (3y - x)^2$

i.e., $5x^2 + 5y^2 - 20x + 10y = 9y^2 - 6xy + x^2$

i.e., $4x^2 + 6xy - 4y^2 - 20x + 10y = 0$

In this 2nd degree equation representing a pair of lines, the sum of the coefficients of x^2 and $y^2 = 4 + (-4) = 0$;

So these two lines are at right angles.

Hence the pair of tangents from the point (1, 2) to the circle are mutually perpendicular.

Ans. to Q. No. 10: We know that the equation of the chord of contact of the point (x_1, y_1) w.r. to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is,

$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0.$

Hence the equation of the chord of contact of the point (4, 3) w.r. to the circle $x^2 + y^2 - 3x + 12y + 6 = 0$ is,

$$x.4 + y.3 - \frac{3}{2}(x+4) + \frac{12}{2}(y+3) + 6 = 0$$

i.e., $8x + 6y - 3x - 12 + 12y + 36 + 12 = 0$

i.e., $5x + 18y + 36 = 0.$



4.11 MODEL QUESTIONS

Q.1: Find the equation of a circle with centre (4, –6) and radius 7.

Q.2: Find the centre and radius of the circle $x^2 + y^2 + 4x - 6y = 0$.

Q.3: Find the equation of the circle which passes through the points (1, 0), (–1, 0) and (0, 1).

Q.4: Find the equation of the tangent and normal to the given circle at the indicated points.

a) $x^2 + y^2 = 25$; (–3, –4)

b) $x^2 + y^2 - 2x - 10y + 1 = 0$; (–3, 2)

Q.5: Find the normal to the circle, $x^2 + y^2 – 3x + 10y = 15$ at (4, –11).

Q.6: Find the equation of the tangent at the point (7, 2) to the circle, $x^2 + y^2 – 26x + 12y + 105 = 0$.

Q.7: Find the equation of the tangent and normal to the circle, $2x^2 + 2y^2 + 3x – 4y + 1 = 0$ at (–1, 2).

Q.8: Is the point (7, –11) inside or outside the circle $x^2 + y^2 = 10x$?

Q.9: Find the equation of the tangents from the point (0, 1) to the circle, $x^2 + y^2 – 2x – 6y + 6 = 0$.

Q.10: Find the equation of the chord of contact of tangents from the point (4, 1) to the circle, $x^2 + y^2 –10x + y + 21 = 0$.

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UNIT 5: CONIC SECTIONS

UNIT STUCTURE

5.1 Learning Objectives

5.2 Introduction

5.3 Parabola and its Standard Equation

5.4 Ellipse and its Standard Equation

5.5 Hyperbola and its Standard Equation

5.6 General Equation of the Second Degree

5.7 Let Us Sum Up

5.8 Further Reading

5.9 Answers to Check Your Progress

5.10 Model Questions

5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know parabola and its equation
- know ellipse and its equation
- know hyperbola and its equation
- know general equation of the second degree.

5.2 INTRODUCTION

If a point moves in a plane such that its distance from a fixed point in the plane always bears a constant ratio to its (perpendicular) distance from a fixed straight line in the plane (which does not pass through the fixed point), then the locus of the moving point is called a conic section or simply a conic. The fixed point is called the focus, the fixed line is called the directrix and the constant ratio, usually denoted by e, is called the eccentricity of the conic. e, being the ratio of two lengths, is a pure positive real number.

The conic is called a parabola, ellipse or hyperbola according as $e = 1$, $e < 1$ or $e > 1$ respectively. These curves are called conic sections because they can be realised as intersections of a double-napped right

circular cone by a plane. The plane section of such a cone may also be a circle or a pair of straight lines which are called improper conics.

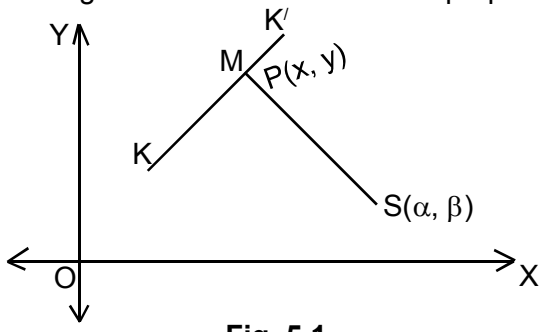


Fig. 5.1

Referred to arbitrarily chosen rectangular coordinate axes, let S(α, β) be the focus and the line KK/ : Ax + By + C = 0 be the directrix of a conic whose eccentricity is e. Let P(x, y) be any point on the conic and PM $\perp$ KK/ . By definition of the conic, we have,

$$\frac{PS}{PM} = e$$

$$\Rightarrow PS^2 = e^2.PM^2$$

$$\Rightarrow (x-\alpha)^2 + (y-\beta)^2 = e^2\left(\frac{Ax+By+C}{\sqrt{A^2+B^2}}\right)^2$$

As this equation is satisfied by the coordinates of every point P on the conic, this is the equation of the conic. On simplication the equation takes the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ which is the most general equation of the second degree in x and y.

5.3 PARABOLA AND ITS STANDARD EQUATION

a) Parabola: A conic whose eccentriaty is 1 is called a parabola. Hence a parabola is the locus of a point which moves in a plane such that its distance from a fixed point (called the focus) is always equal to its distance from a fixed straight line (called the directix).

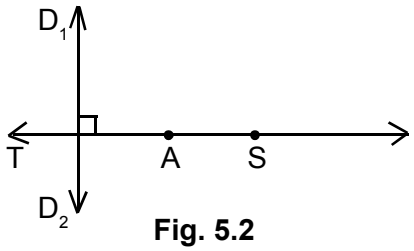


Fig. 5.2

Let S be the focus and D_1D_2 be the directrix of a parabola. The line passing through the focus S and perpendicular to the directrix D_1D_2 is called the axis of the parabola. Suppose the axis intersects the directrix at T. Then if A be the middle point of the line segment TS, then AS = AT i.e., the point A is equidistant from the focus and the directrix, and hence A must be a point on the parabola. This particular point A on the axis of the parabola is known as the vertex of the parabola.

b) Standard Equation of a Parabola:

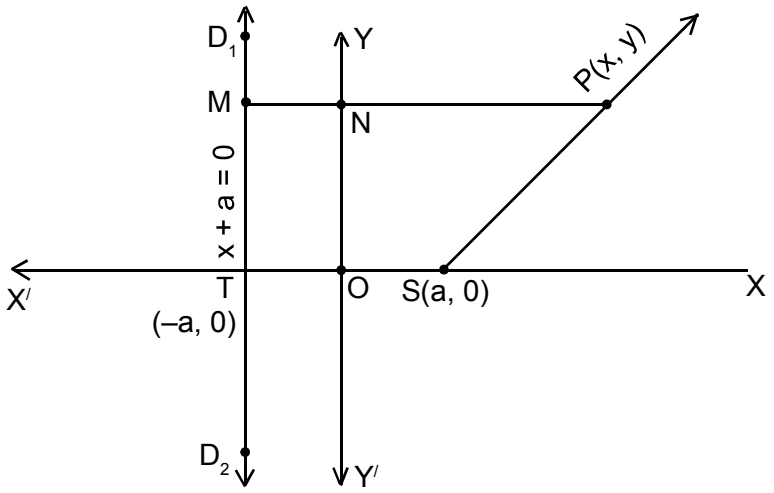


Fig. 5.3

Let S be the focus and D_1D_2 be the directrix of a parabola. Draw ST perpendicular to the directrix D_1D_2. Let O be the middle point of TS; then, as OS = OT, O lies on the parabola. By definition the line TS is the axis and the point O is the vertex of the parabola.

Take the point O as the origin, x-axis along the line OS and y-axis along the line perpendicular to OS at O, as shown in the above figure. Let OS = OT = a, a being a positive constant. Then S has co-ordinates (a, 0), T has co-ordinates (–a, 0) and the equation of the directrix D_1D_2, which is a line through T(–a, 0) parallel to the y-axis, is x = –a, i.e., x + a = 0.

Let P(x, y) be any point on the parabola and PM be drawn perpendicular to the directrix D_1D_2. Then, by definition of the parabola, we have, PS = PM.

Now, PS = $\sqrt{(x-a)^2+y^2}$,

and, if PM intersects y -axis at N, we have

PM = PN + NM = PN + OT = x + a.

Hence we get, $\sqrt{(x-a)^2+y^2}$ = x + a,

Whence, on squaring both sides, we get,

$(x-a)^2 + y^2 = (x+a)^2$

i.e., $y^2 = (x+a)^2-(x-a)^2$

i.e., $y^2 = 4ax$.

This is the equation of the parabola, known as its standard equation.

Remarks: The equation of a parabola can be put in the simplest form $y^2 = 4ax$ ($a > 0$), known as the standard equation, by a proper choice of rectangular coordinate axes. This is the equation of a parabola referred to its vertex as the origin and its axis as the axis of x.

c) Shape of a Parabola: Taking the standard equation $y^2 = 4ax$ ($a > 0$) of a parabola, we observe the following.

i) The curve passes through the origin.

ii) When $x < 0$, the corresponding values of y become imaginary. So no part of the curve lies to the left of y-axis.

iii) For each $x > 0$, there are two numerically equal and opposite values of y. $y = \pm 2\sqrt{ax}$. Hence the curve is symmetical about its axis.

iv) For x = 0, we get $y^2 = 0 \Rightarrow y = 0, 0$. Thus the y-axis meets the parabola $y^2 = 4ax$ at two points coincident at the origin, i.e. the y-axis is the tangent to the parabola $y^2 = 4ax$ at the origin.

v) As x increases, $y = \pm 2\sqrt{ax}$ increases numerically and $|y| \to \infty$ as $x \to \infty$. It follows that the parabola is not a closed curve and it extends upto infinity on the right side of y-axis both above and below the x-axis.

Taking all the above facts into consideration the shape of the parabola $y^2 = 4ax$ ($a > 0$) is found to be as drawn below:

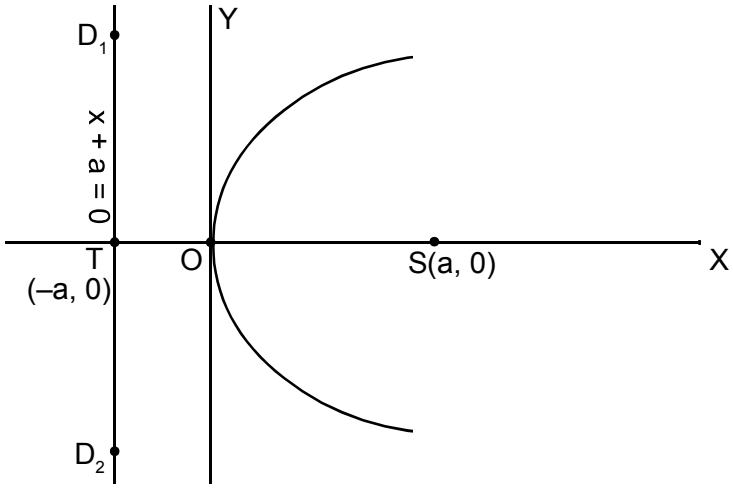


Fig. 5.4

This parabola opens to the right.

d) Parabolas Represented by $y^2 = -4ax$, $x^2 = 4ay$ and $x^2 = -4ay$, (a >0):

The equation $y^2 = -4ax$ $(a > 0)$ represents a parabola with vertex at the origin and the focus at the point $(-a, 0)$, the directrix being the line $x - a = 0$. Its axis is the x-axis and it opens to the left.

The equation $x^2 = 4ay$ $(a > 0)$ represents a parabola with vertex at the origin and focus at the poin $(0, a)$, the equation of the directrix being $y + a = 0$; it opens upward. Lastly the equation $x^2 = -4ay$ represents a parabola with vertex at the origin and focus at $(0, -a)$, the directrix being the line $y - a = 0$. In either case the y-axis is the axis of the parabola. While the parabola $x^2 = 4ay$ opens upward, the parabola $x^2 = -4ay$ opens downward.

These parabolas are shown below:

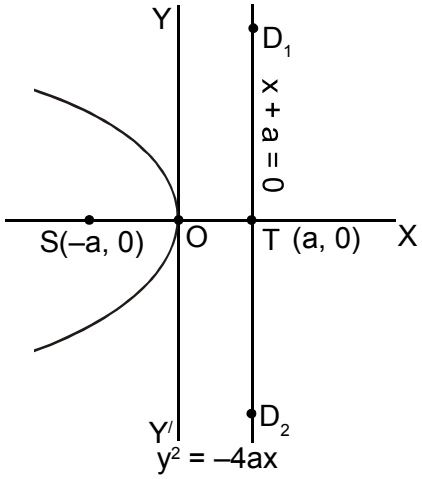


Fig. 5.5

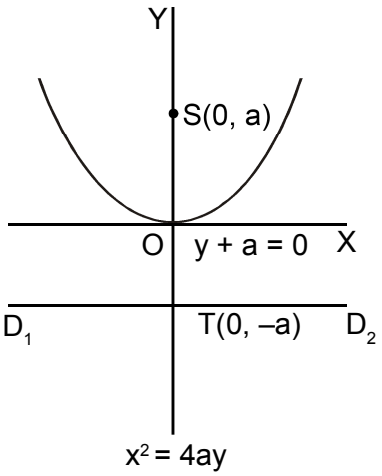


Fig. 5.6

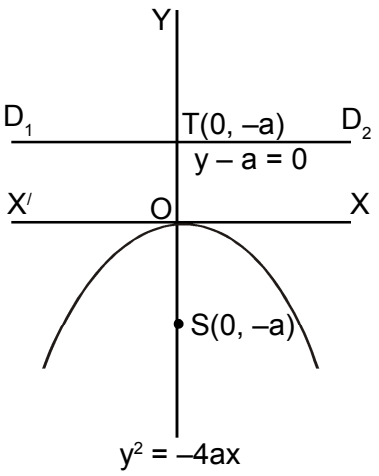


Fig. 5.7

e) Focal Distance, Focal Chord and Latus Rectum: The distance of any point of a parabola from its focus is called the focal distance of the point.

A chord of a parabola passing through its focus is called a focal chord of the parabola.

The focal chord of a parabola perpendicular to its axis is called the latus rectum of the parabola. In other words the chord of a parabola through its focus parallel to its directrix (i.e. perpendicular to its axis) is called the latus rectum of the parabola.

Consider the parabola $y^2 = 4ax$. In the adjacent figures is its focus and LSL' is its latus rectum. As the parabola is symmetric about the x-axis, SL and SL' are equal in length, say l. Then L and L' have coordinates (a, l) and (a, –l) respectively. As L (a, l) is a point on the parabola $y^2 = 4ax$, we get,

$l^2 = 4a.a = 4a^2 \Rightarrow l = 2a$.

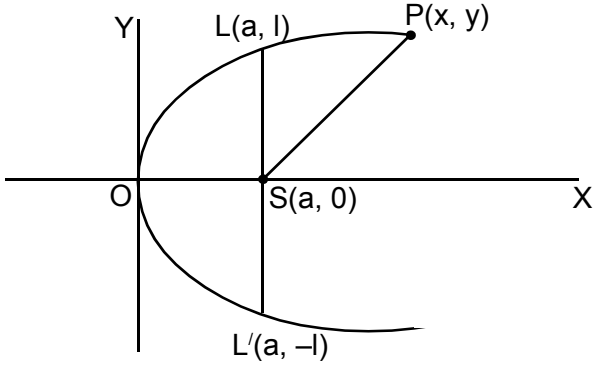


Fig. 5.8

$\therefore$ Length of the latus rectum $LSL' = 2SL = 2l = 4a$.

The end points of the latus rectum are L(a, 2a) and L'(a, –2a).

The latus rectum is parallel to y-axis and passes through S(a, o); so its equation is x = a.

If P(x, y) is any point on the parabola, the focal distance of the point P

$= PS = \sqrt{(x-a)^2 + (y-0)^2} = \sqrt{(x-a)^2 + y^2}$

$= \sqrt{(x-a)^2 + 4ax} = \sqrt{(x+a)^2} = x + a.$

It is easily seen that the length of the latus rectum of each of the parabolas $y^2 = -4ax$, $x^2 = 4ay$ and $x^2 = -4ay$ $(a > 0)$ is also 4a.

The main results obtained in 4.2 are now summarised below (a > 0):

Equation of Parabola	Focus	Vertex	Equation of Directrix	Equation of axis	Length of latus rectum	Equation of latus rectum
$y^2 = 4ax$	(a, 0)	(0, 0)	x+a = 0	y=0 (x-axis)	4a	x–a = 0
$y^2 = –4ax$	(–a, 0)	(0, 0)	x–a = 0	y = 0 (x-axis)	4a	x+a = 0
$y^2 = 4ay$	(0, a)	(0, 0)	y+a = 0	x = 0 (y-axis)	4a	y–a = 0
$x^2 = –4ay$	(0, –a)	(0, 0)	y–a = 0	x = 0 (y-axis)	4a	y+a = 0

Example 1: Find the vertex, focus, directrix and length of the latus rectum of the following parabolas: (i) $y^2 = 10x$, (ii) $5x^2 + 24y = 0$

Solution:

i) The equation of the parabola is $y^2 = 10x$ which is of the form $y^2 = 4ax$ when 4a = 10, i.e., $a = \frac{10}{4} = \frac{5}{2}$. Its vertex is (0, 0), focus is (a, 0), i.e., $\left(\frac{5}{2}, 0\right)$, directrix has equation x + a = 0 i.e. $x + \frac{5}{2} = 0$, and length of its latus rectum $= 4a = 4 \times \frac{5}{2} = 10$.

ii) The equation of the parabola is $5x^2 + 24y = 0$, i.e., $5x^2 = –24y$, i.e., $x^2 = -\frac{24}{5}y$ which is of the form $x^2 = – 4ay$, where $4a = \frac{24}{5}$, i.e., $a = \frac{6}{5}$.

Its vertex is (0, 0), focus is (0, –a), i.e., $\left(0, -\frac{6}{5}\right)$, diretrix is the line y – a = 0, i.e., $y - \frac{6}{5} = 0$, i.e., 5y – 6 = 0, and length of its latus rectum $= 4a = \frac{24}{5}$.

Example 2: Show that the equation $4y^2 - 20x - 8y + 39 = 0$ represents a parabola. Find its vertex, focus, axis and latus rectum.

Solution:

$4y^2 - 20x - 8y + 39 = 0$

$\Rightarrow 4y^2 - 8y = 20x - 39$

$\Rightarrow y^2 - 2y = 5x - \frac{39}{4}$

$\Rightarrow y^2 - 2y + 1 = 5x + 1 - \frac{39}{4} = 5x - \frac{35}{4}$

$\Rightarrow (y - 1)^2 = 5\left(x - \frac{7}{4}\right)$

Transfering the origin to the point $\left(\frac{7}{4}, 1\right)$, the above equation becomes $Y^2 = 5X$.

Which represents a parabola. Hence the given equation $4y^2 - 20x - 8y + 39 = 0$ represents a parabola.

Refered to the parallel axes through the new origin, the equation of the parabola is $Y^2 = 5X$ and its vertex is (0, 0), focus is $\left(\frac{5}{4}, 0\right)$ ($\because$ $4a = 5$) and axis is the line $Y = 0$ (new x-axis). Hence refered to the original axes, the vertex is the point $\left(\frac{7}{4}, 1\right)$, the focus is the point (3, 1). $\left(\because x = X + \frac{7}{4},\ y = Y + 1\right)$, and the axis is the line $y = 1$.

Also latus rectum of the parabola $= 4a = 5$.

Note: This technique is applicable to equation of the form $y^2 + ax + by + c = 0$ and $x^2 + ax + by + c = 0$.



CHECK YOUR PROGRESS

Q1: Find the coordinates of the focus and the vertex, the equation of the directrix and the axis, and the length of the latus rectum of each of the following parabolas:

i) $3y^2 = 8x$ ii) $y^2 = -8x$ iii) $x^2 = 10y$

iv) $x^2 = -8y$ v) $x^2 - 6x + 4y + 1 = 0$

Q.2: Find the equation of the parabola with focus (4, 0) and the directrix x = –4.

Q.3: Find the equation of the parabola with vertex at the origin, focus on the positive side of y-axis, and having 20 as the length of the latus rectum.

5.4 ELLIPSE AND ITS STANDARD EQUATION

a) Ellipse: An ellipse is a conic whose eccentricity is less than 1. Thus an ellipse is the locus of a point which moves in a plane such that its distance from a fixed point bears a constant ratio e < 1 to its perpendicular distance from a fixed line. The fixed point is the focus, the fixed line is the directrix and the fixed ratio denoted by e. where e < 1, is the eccentricity of the ellipse.

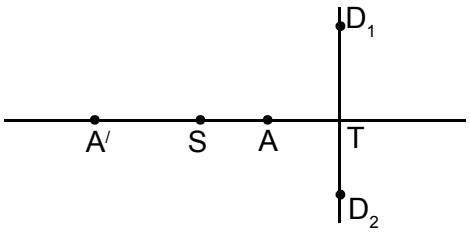


Fig. 5.9

Let S be the focus, D_1D_2 the directrix and e(< 1) the eccentricity of an ellipse. Let the perpendicular from S to D_1D_2 intersect it at T. We can find a point A on ST between S and T such that AS : AT = e : 1. Clearly A lies on the ellipse. Also AS = e. AT < AT as e < 1.

As e < 1, there is another point $A^/$ on TS produced such that $A^/S : A^/T = e : 1$ and so $A^/$ also lies on the ellipse.

Thus on the line drawn through the focus perpendicular to the directrix of the ellipse there exist two points which divide the segment intercepted between the focus and directrix internally and externally in the same ratio e : 1. These two special points of the ellipse are called its vertices and the line segment joining them is called the major axis of the ellipse.

b) Standard Equation of an Ellipse:

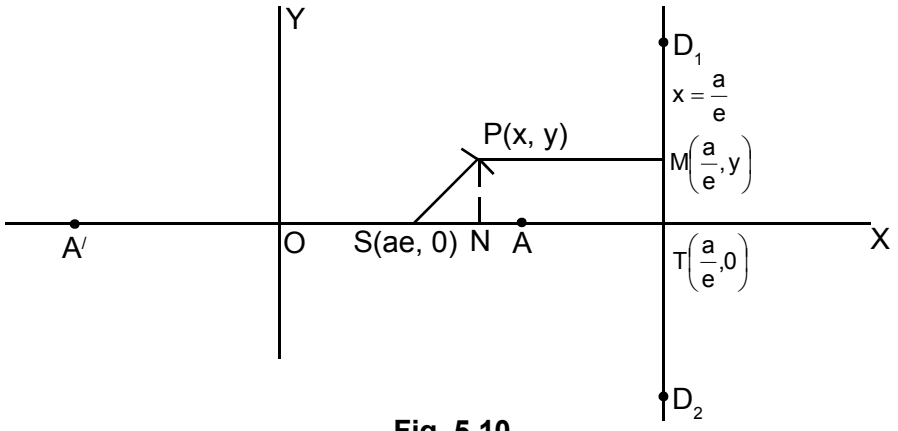


Fig. 5.10

Let S be the focus, D_1D_2 the directrix and e(< 1) be the eccentricity of an ellipse. Let the line drawn through S perpendicular to D_1D_2 intersect it at T. Then there exist two points A and A/ on this line which divide the segment ST internally and externally respectively in the same ratio e : 1. As AS/AT = e and A/S / A/T = e, A and A/ are two points on the ellipse. Let O be the middle point of A/A and let A/ O = OA = a so that A/A = 2a. Then,

AS + A/S = e (AT + A/T)

$\Rightarrow$ A/A = e (OT – OA + A/O + OT)

$\Rightarrow$ 2a = e (OT – a + a + OT) = e. 20T

$\Rightarrow$ OT = $\frac{a}{e}$,

and A/S – AS = e(A/T– AT)

$\Rightarrow$ (A/O + OS) – (OA – OS) = e.A/A

$\Rightarrow$ 20S = e.2a $\Rightarrow$ OS = ae.

Thus OS = ae and OT = $\frac{a}{e}$.

Let us take O as the origin, x-axis along OA and y-axis along the line through O $\perp$ to A/A (i.e. parallel to the directrix D_1D_2) as shown in the above figure. Then the focus S has coordinates (ae, o) and the equation of the direxitrix D_1D_2, which is parallel to y-axis and passes through $T\left(\frac{a}{e}, 0\right)$ is $x = \frac{a}{e}$.

Let P(x, y) be any point on the ellipse. Join PS and draw PM perpendicular to the directrix D_1D_2 and PN perpendicular to the x-axis. Then, TM = NP = y and so the coordinates of M are $\left(\frac{a}{e}, y\right)$.

Now by definition of ellipse we have,

$$\frac{PS}{PM} = e$$

$$\Rightarrow PS^2 = e^2 \cdot PM^2$$

$$\Rightarrow (x - ae)^2 + y^2 = e^2\left[\left(\frac{a}{e} - x\right)^2 + (y - y)^2\right]$$

$$\Rightarrow x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2$$

$$\Rightarrow (1 - e^2)x^2 + y^2 = a^2(1 - e^2)$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

As $0 < e < 1$, $a^2(1 - e^2)$ is a positive constant; let it be denoted by b^2 where b is chosen to be positive. Then the above relation becomes,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the required equation of the ellipse known as its standard equation.

Note that $b^2 = a^2(1 - e^2) < a^2$, i.e. $0 < b < a$

c) Shape of an Ellipse, Existence of Second Focus and Second Directrix: Consider the ellipse with standard equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b > 0).$$

We observe the following points:

i) If (x, y) lies on the curve, the points (–x, y), (x, –y) and (–x, –y) also lie on the curve. So the curve is symmetrical about both the coordinate axes and the origin.

ii) The curve meets the x-axis at A(a, 0) and A′(–a, 0) and it meets the y-axis at B(0, b) and B′(0 – b).

iii) From the equation of the ellipse we get,

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}, \quad x = \pm\frac{a}{b}\sqrt{b^2 - y^2}$$

Clearly y is real iff $-a \le x \le a$ and x is real iff $-b \le y \le b$. Hence the curve is a closed one lying entirely inside the rectangle formed by the lines x = –a, x = a, y = b, y = –b.

The lines x = a and x = –a are two tangents of the curve at A(a, 0) and A′(–a, 0) parallel to the y-axis while the lines y = b and y = –b are two tangents to the curve at B(0, b) and B′(0, –b) respectivity parallel to the x-axis.

Taking the above points into consideration the shape of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b > 0) is found to be as under:

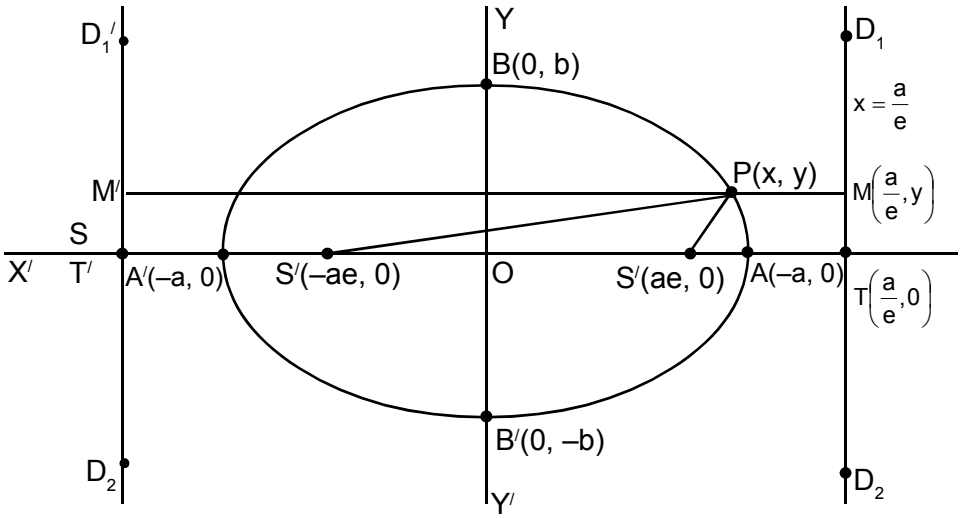


Fig. 5.11

The points A and A′ are called the vertices of the ellipse. The line AA′ is called the major axis and the line BB′ is called the minor axis of the ellipse. Their point of intersection O, which is the middle point of either axis of the ellipse, bisects every chord through it and is referred to as the centre of the ellipse. The length of the major axis of the above ellipse is 2a and that of the minor axis is 2b.

On the negative side of the x-axis mark two points S′ and T′ such than OS′ = OS and OT′ = OT; then S′ and T′ have coordinates (–ae, 0) and $\left(-\frac{a}{e}, 0\right)$ respectively. Also draw through T′$\left(-\frac{a}{e}, 0\right)$ the line

$D_1'D_2'$ perpendicular to x-axis; then the equation of the line $D_1'D_2'$ is $x = -\frac{a}{e}$. Let P(x, y) be any point on the ellipse whose focus is S(ae, o) and directrix is D_1D_2. Through P(x, y) draw a line parallel to the x-axis (i.e. major axis) intersecting D_1D_2 at M and $D_1'D_2'$ at M'. Clearly the points M, M' and P have the same ordinate y. Thus M and M' have coordinates $\left(\frac{a}{e}, y\right)$ and $\left(-\frac{a}{e}, y\right)$ respectively. By definition of the ellipse, we have,

$$\frac{PS}{PM} = e$$

$$\Rightarrow PS^2 = e^2.PM^2$$

$$\Rightarrow (x-ae)^2 + (y-o)^2 = e^2\left[\left(x-\frac{a}{e}\right)^2 + (y-y)^2\right]$$

$$\Rightarrow (x-ae)^2 + y^2 = (ex-a)^2$$

$$\Rightarrow (x+ae)^2 - 4aex + y^2 = (ex+a)^2 - 4aex$$

$$\Rightarrow (x+ae)^2 + y^2 = (ex+a)^2 = e^2\left(x+\frac{a}{e}\right)^2$$

$$\Rightarrow (x+ae)^2 + (y-o)^2 = e^2\left[\left(x+\frac{a}{e}\right)^2 + (y-y)^2\right]$$

$$\Rightarrow PS'^2 = e^2.\ PM'^2$$

$$\Rightarrow \frac{PS'}{PM'} = e$$

Hence the same ellipse would be described by the point P if S'(–ae, 0) is taken as the focus and the line $D_1'D_2'$, whose equation is $x = \frac{a}{e}$, is taken as the directrix. Thus the ellipse has a second focus S'(–ae, 0) and second directrix $D_1'D_2'$ represented by $x = -\frac{a}{e}$.

An Important Note: We have introduced vertices, major axis, minor axis and centre of an ellipse by considering its equation in the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is only a matter of convenience and

these notions have nothing to do with any particular coordinate system. We can define them independent of any coordinate system as follows:

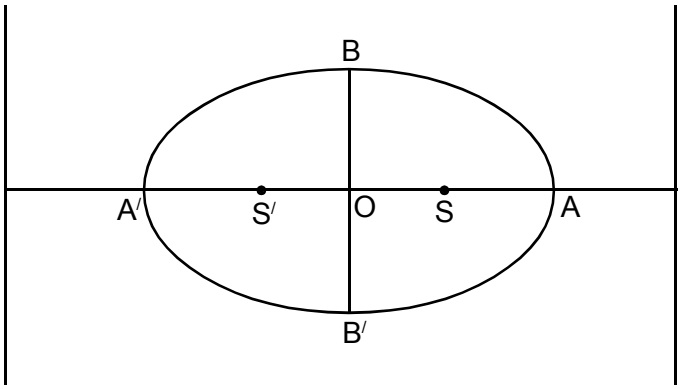


Fig. 5.12

i) The line through the two foci S and S/ of an ellipse is perpendicular to the two directrices of the ellipse and this line intersects the ellipse in two points, say A and A/. The line segment A/A is called the major axis of the ellipse. The end points of the major axis are called the vertices of the ellipse.

ii) The line through the middle point O of the major axis at right angles to it intersects the ellipse in two points, say B and B/. The line segment B/B is called the minor axis of the ellipse. O is also the middle point of BB/.

iii) The point of intersection, O, of the major and minor axis, which is the middle point of each axis, is called the centre of the ellipse. Every chord of the ellipse through its centre is bisected at that point.

It is to be noted that the equation of an ellipse in its standard from $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b > 0) is obtained by choosing its centre as the origin, the x-axis along its major axis and the y-axis along its minor axis, where 2a is the length of its major axis and 2b is the length of its minor axis. a is called the length of the semi-major axis and b the length of the semi-minor axis of the ellipse. Its eccentricity, e, is given by $b^2 = a^2(1 – e^2)$.

Whence $e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{c}{a}$, 2c being the distance between the two foci.

It is easy to see that the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where b>a>o represents an ellipse as shown in the adjacent figure. Its vertices are B(0, b) and B$'$(0, –b), foci are S(0, be) and S$'$(0, –be), major axis BB$'$ = 2b lies on the y-axis, minor axis AA$'$ = 2a lies on the x-axis and its centre is the origin. The eccenticity 'e' of the ellipse is connected with its semi-major axis b and semi-minor axis a by the relation $a^2 = b^2(1 - e^2)$.

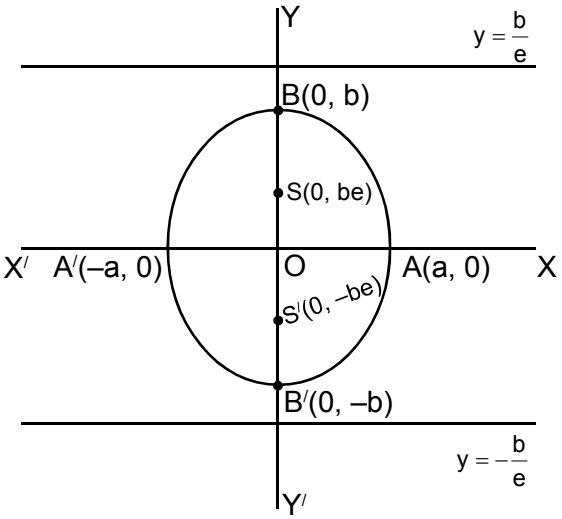


Fig. 5.13

Whence $e = \frac{\sqrt{b^2 - a^2}}{b} = \frac{c}{b}$, 2c being the distance between the two foci.

d) Focal Chord and Latus Rectum: Focal chord of an ellipse is a chord of it which passes through a focus. In the adjoining figure PSQ and P$'$ S$'$ Q$'$ are two focal chords of the ellipse.

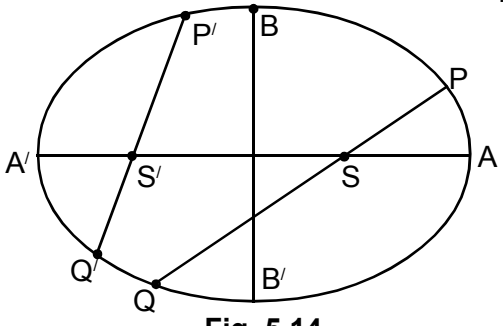


Fig. 5.14

The **latus rectum** of an ellipse is its focal chord perpendicular to the major axis. Clearly an ellipse has two latera recta through its two foci and they are of the same length. Each latus rectum is bisected at the corresponding focus and each half is called a semi-latus rectum.

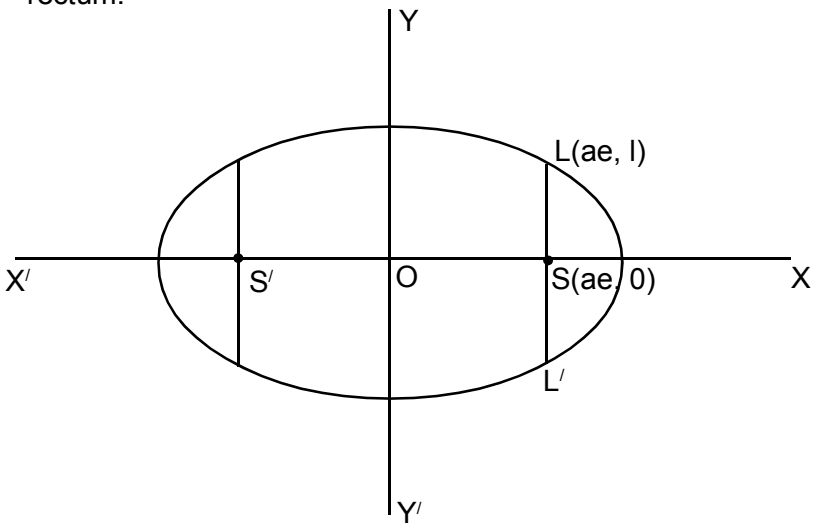


Fig. 5.15

Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where a > b > o. Let LSL/ be its latus recum through the focus S(ae, o) and let SL = l. Then the point L(ae, l) lies on the given ellipse. So we have

$$\frac{(ae)^2}{a^2}+\frac{l^2}{b^2}=1$$

$$\Rightarrow \frac{l^2}{b^2}=1-e^2=\frac{1}{a^2}.a^2\left(1-e^2\right)=\frac{b^2}{a^2}$$

$$\Rightarrow l^2=\frac{b^4}{a^2}.$$

$$\Rightarrow l=\frac{b^2}{a}$$

Hence the length of semi-latus rectum = $l=\frac{b^2}{a}$

and the length of latus rectum = $2l=\frac{2b^2}{a}$.

The equations of the two latera recta are, x = ±ae.

Likewise it is easily seen that in case of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (b > a > 0), whose major axis lies on y-axis and is of length 2b, the length of semi-latus rectum is $\frac{a^2}{b}$, length of latus rectum is $\frac{2a^2}{b}$ and the equations of the latera recta are y = ±be.

SUMMARY OF RESULTS:

Equation of ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b > 0	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, b > a > 0
Centre	(0, 0)	(0, 0)
Vertices	(±a, 0)	(0, ±b)
Eccentricity	$e=\frac{\sqrt{a^2-b^2}}{a}$	$e=\frac{\sqrt{b^2-a^2}}{b}$
Foci	(±ae, 0)	(0, ±be)
Equations of the directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Length of the major axis	2a	2b
Length of the minor axis	2b	2a
Equation of the major axis	y = 0 (x-axis)	x = 0 (y-axis)
Equation of the minor axis	x = 0 (y-axis)	y = 0 (x-axis)
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equations of the latera recta	x = ±ae	y = ±be

Example 1: Find the equation of an ellipse referred to its axes as coordinate axes if its eccentricity is $\frac{1}{4}$ and foci are $\left(\pm\sqrt{2}, 0\right)$.

Solution: Let the required equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

As the eccentricity is $\frac{1}{4}$ and foci are $(\pm\sqrt{21}, 0)$, we have,

$e=\frac{1}{4}$ and $ae=\sqrt{2}$.

$\therefore\ a=\frac{ae}{e}=\frac{\sqrt{2}}{\frac{1}{4}}=4\sqrt{2}$

and $b = a\sqrt{1-e^2}=4\sqrt{2}\sqrt{1-\left(\frac{1}{4}\right)^2}$

$= 4\sqrt{2}\times\frac{\sqrt{15}}{4}=\sqrt{30}$

$\therefore$ The required equation of the ellipse is $\frac{x^2}{\left(4\sqrt{2}\right)^2}+\frac{y^2}{\left(\sqrt{30}\right)^2}=1$

i.e., $\frac{x^2}{32}+\frac{y^2}{30}=1$

Example 2: Find the foci, equations of the directrices and the length of the latus rectum of the ellipse $25x^2 + 4y^2 = 100$.

Solution: The equation of the given ellipse is,

$25x^2 + 4y^2 = 100$

i.e., $\frac{x^2}{4}+\frac{y^2}{25}=1$

i.e., $\frac{x^2}{2^2}+\frac{y^2}{5^2}=1$

Which is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with a = 2 and b = 5, As b > a, the major axis lies on the y-axis and the minor axis on x-axis.

Hence $e=\frac{\sqrt{b^2-a^2}}{b}=\frac{\sqrt{5^2-2^2}}{5}=\frac{\sqrt{21}}{5}$.

$\therefore$ The coordinates of the foci are (0, ±be) i.e., $(0, \pm\sqrt{21})$ and the equations of the directrices are $y=\pm\frac{b}{e}$, i.e., $y=\pm\frac{25}{\sqrt{21}}$

Also the length of the latus rectum = $\frac{2a^2}{b} = \frac{2\times 2^2}{5} = \frac{8}{5}$

Example 3: Show that the equation $3x^2 + 4y^2 + 6x - 8y = 5$ represents an ellipse. Find its centre, eccentricity and latus rectum.

Solution: $3x^2 + 4y^2 + 6x - 8y = 5$

$\Rightarrow 3(x^2 + 2x) + 4(y^2 - 2y) = 5$

$\Rightarrow 3(x^2 + 2x + 1) + 4(y^2 - 2y + 1) = 5 + 3 + 4$

(Completing the perfect squares)

$\Rightarrow 3(x + 1)^2 + 4(y - 1)^2 = 12$

$\Rightarrow \frac{(x+1)^2}{4} + \frac{(y-1)^2}{3} = 1$

Transferring to parallel axes through (–1, 1), the above equation transforms to $\frac{x^2}{2^2} + \frac{y^2}{(\sqrt{3})^2} = 1$

Which represents an ellipse whose centre is the new origin whose old coordinates are (–1, 1).

Here, a = 2, b = $\sqrt{3}$, So that a > b.

$\therefore\ e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{4-3}}{2} = \frac{1}{2}$

Also the length of latus sectum = $\frac{2b^2}{a} = \frac{2\times 3}{2} = 3$

Hence the given equation represents an ellipse whose centre is the point (–1, 1), eccentricity is $\frac{1}{2}$ and the latus rectum is 3.



CHECK YOUR PROGRESS

Q.4: Find the lengths of the axes, the eccentricity, the coordinates of the vertices and of the foci, the length of the latus rectum and the equations of the directrices of each of the following ellipses:

i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ ii) $3x^2 + 2y^2 = 18$

Q.5: Find the equation of the ellipse whose centre is at the origin, major axis lies on the x-axis, eccentricity is $\frac{2}{3}$ and latus rectum has length 5 units.

5.5 HYPERBOLA AND ITS STANDARD EQUATION

a) Hyperbola: A hyperbola is a conic whose eccentricity is greater than unity. In other words a hyperbola is the locus of a point which moves in a plane in such a way that its distance from a fixed point bears a constant ratio e > 1 to its perpendicular distance from a fixed line. The fixed point is the focus, the fixed line is the directrix and the fixed ratio e > 1 is the eccentricity of the hyperbola.

b) Standard Equation of a Hyperbola:

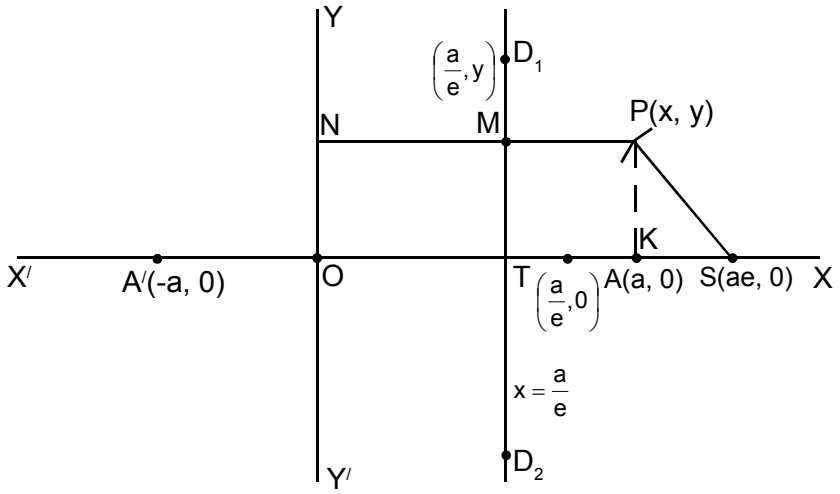


Fig. 5.16

Let S be the focus and D_1D_2 be the directrix of a hyperbola whose eccentricity is e. Draw a line through S perpendicular to D_1D_2 intersecting it at T. Let A be a point on this line between S and T such that AS/AT = e; then A lies on the hyperbola. As e > 1, there exists another point A' on ST produced such that $A'S / A'T = e$; this point A' also lies on the hyperbola. These two points A and A' of the hyperbola are called its vertices and the middle point O of the segment $A'A$ is called its centre.

Let us now take O as the origin, x-axis along OS and y-axis along the line through O perpendicular to OS. Let, $A'A = 2a$. Then the coordinates of A and A' are (a, 0) and (–a, 0) respectively. We have,

$AS + A'S = e.AT + e.A'T = e(AT + A'T)$

$\Rightarrow (OS - OA) + (A'O + OS) = e.\ AA'$

$\Rightarrow 20S = e.2a$

$\Rightarrow OS = ae$

and $A'S - AS = e.\ A'T - e.AT = e(A'T - AT)$

$\Rightarrow A'A = e.\{(A'O + OT) - (OA - OT)\} = e.2OT$

$\Rightarrow 2a = e.2OT$

$\Rightarrow OT = \frac{a}{e}$

As S and T lie on the x-axis we see that their coordinates are (ae, 0) and $\left(\frac{a}{e}, 0\right)$ respectively. The directrix D_1D_2 is a line parallel to the y-axis and it passes through $T\left(\frac{a}{e}, 0\right)$, so its equation is $x = \frac{a}{e}$.

Let P(x, y) be any point on the hyperbola. Join PS and draw PM perpendicular to the directrix D_1D_2; when produced PM meets the y-axis at N perpendicularly. Draw $PK \perp OX$. Then TM = KP = y is the ordinate of M and its abscissa is $OT = \frac{a}{e}$. So M has coordinates $\left(\frac{a}{e}, y\right)$. As P is a point on the hyperbola, we have, by definition

$$\frac{PS}{PM} = e$$

$\Rightarrow PS^2 = e^2.PM^2$

$$\Rightarrow (x - ae)^2 + (y - 0)^2 = e^2\left[\left(x - \frac{a}{e}\right)^2 + (y - y)^2\right]$$

$\Rightarrow (x–ae)^2 + y^2 = (ex–a)^2$

$\Rightarrow x^2 + a^2e^2 – 2aex + y^2 = e^2x^2 + a^2 – 2aex$

$\Rightarrow (e^2–1)x^2 – y^2 = a^2(e^2–1)$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

As e > 1, $a^2(e^2 - 1)$ is a positive constant; let it be denoted by b^2, where b is chosen positive. Then the above equation takes the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Which is the required equation of the hyperbola known as its standard equation.

c) Shape of a Hyperbola; Existence of Second Focus and Second Directrix: To sketch the curve represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we make the following observations:

i) The curve is symmetrical about both the coordinate axes and the origin.

ii) Putting y = 0 in the equation we get x = ±a, so the curve meets the x-axis at A(a, 0) and A′(–a, 0) which are the vertices of the hyperbola. The middle point of AA′, which has been chosen as the origin, is the centre of the hyperbola. The segment AA′ = 2a is called the transverse axis of the hyperbola. The centre bisects every chord through it, hence the name.

iii) From the equation we have $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$ whence y becomes imaginary for –a < x < a; so there are no points on the curve between the lines x = –a and x = a.

For x = a, both the values of y are zero; so the line x = a is the tangent to the hyperbola at the vertex A(a, 0). Likewise the line x = –a is tangent to the hyperbola at the vertex A′(–a, 0).

Also from the equation we have, $x = \pm\frac{a}{b}\sqrt{b^2 + y^2}$

Which is real for every real value of y. So there is no limitation for the possible values of y although the possible values of x must satisfy $|x| \geq a$ as seen above.

iv) As $x \to \infty$ or, $x \to -\infty$, $y \to \pm\infty$

Taking the above points into consideration we see that the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ consists of two separate branches

(extending to infinity on both positive and negative directions of the transverse axis) as shown in the figure below:

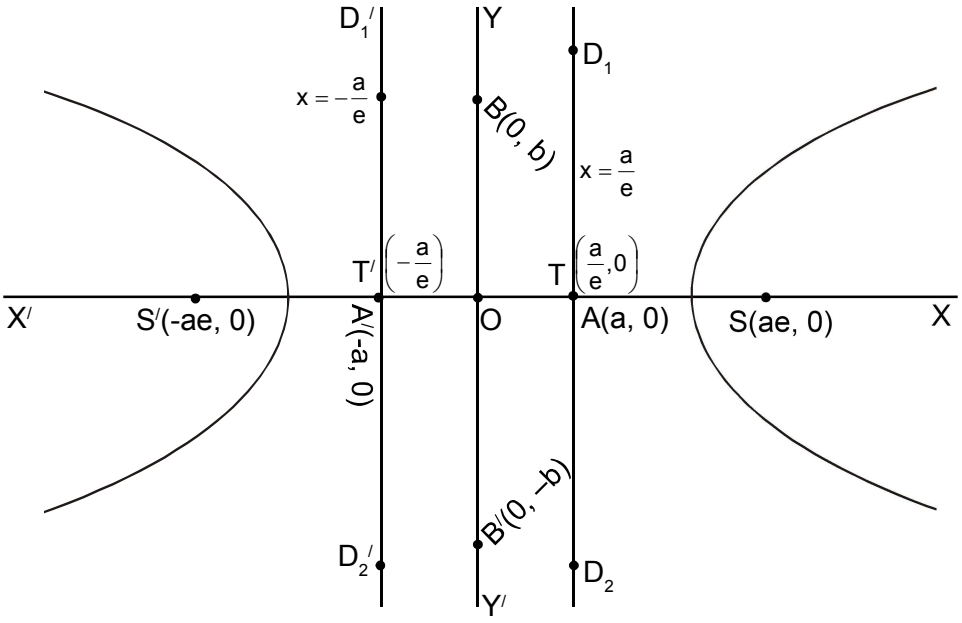


Fig. 5.17

We mark the two points B(0, b) and B/(0, –b) on the y-axis; these points do not lie on the hyperbola. The line segment BB/ = 2b is called the conjugate axis of the hyperbola.

From the symmetry of the hyperbola about its conjugate axis (y-axis) it follows that there must be a second focus S/ situated on the left portion of the transverse axis such that OS/ = OS and a second directrix $D_1/D_2/$ corresponding to S/ cutting OA/ at T/ at right angles such that OT/ = OT. The coordinates of S/ are (–ae, 0) and the equation of $D_1/D_2/$ is $x = -\frac{a}{e}$.

Thus the equation of a hyperbola in its standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is its equation referred to its transverse axis as the x-axis and its conjugate axis as the y-axis, its centre (the point of intersection of the transverse and conjugate axes) being the origin, where 2a is the length of the transverse axis and 2b is the length of the conjugate axis. Its eccentricity e is given by $b^2 = a^2(e^2 - 1)$.

Whence $e = \frac{\sqrt{a^2 + b^2}}{a}$

$AA' = 2a$ and $BB' = 2b$ are the transverse and conjugate axes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

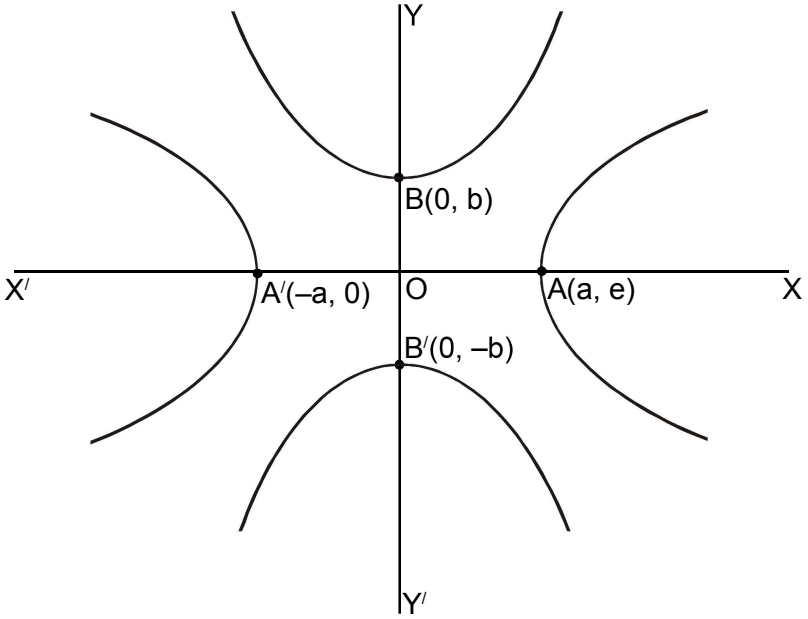


Fig. 5.18

The hyperbola, which has BB' as transverse axis and AA' as conjugate axis, is called the conjugate hyperbola of the original hyperbola. The equation of the conjugate hyperbole is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$; its eccentricity is $e = \frac{\sqrt{a^2 + b^2}}{b}$, its foci are the points $(0, \pm be)$ lying on the y-axis and its directrices are lines parallel to the x-axis having equations $y = \pm \frac{b}{e}$.

d) Latus Rectum: The latus rectum of a hyperbola is its focal chord (i.e. a chord through one of its foci) perpendicular to its transverse axis.

If LSL' be latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ through the focus $S(ae, 0)$ and $SL = SL' = l$, then the pt. $L(ae, l)$ lies on the hyperbola.

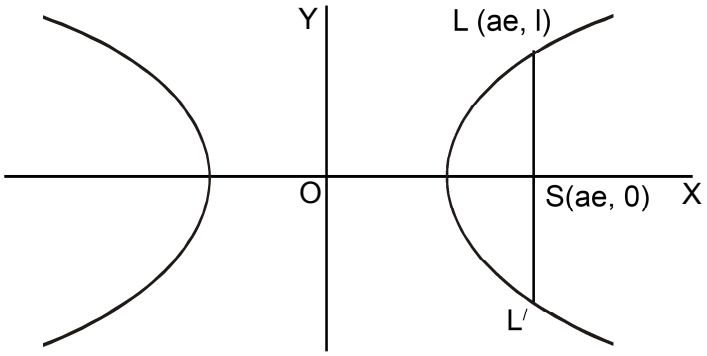


Fig. 5.19

So, $\frac{(ae)^2}{a^2} - \frac{l^2}{b^2} = 1$

$\Rightarrow \quad \frac{l^2}{b^2} = (e^2 - 1) = \frac{a^2(e^2-1)}{a^2} = \frac{b^2}{a^2}$

$\Rightarrow \quad l^2 = \frac{b^4}{a^2}$

$\Rightarrow \quad l = \frac{b^2}{a}$

Hence the length of the semi-latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{b^2}{a}$ and the length of its latus rectum is $\frac{2b^2}{a}$.

It is obvious that the length of the latus rectum of its conjugate hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ is $\frac{2a^2}{b}$.

e) Rectangular Hyperbola: A hyperbola is called a rectangular hyperbola if the lengths of its transverse axis and conjugate axis are equal. Its equation is of the form $x^2 - y^2 = a^2$. Its eccentricity is given by

$$e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{a^2 + a^2}}{a} = \sqrt{2}.$$

Thus the eccentricity of all the rectangular hyperbolas is the same viz. $\sqrt{2}$, whatever be the equal length of its transverse and conjugate axes.

SUMMARY OF RESULTS:

Equation of hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$
Centre	(0, 0)	(0, 0)
Vertices	(±a, 0)	(0, ±b)
Eccentricity	$e=\frac{\sqrt{a^2+b^2}}{a}$	$e=\frac{\sqrt{a^2+b^2}}{b}$
Foci	(±a, 0)	(0, ±b)
Equations of the directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Length of the transverse axis	2a	2b
Length of the conjugate axis	2b	2a
Equation of the transverse axis	y = 0 (x-axis)	x = 0 (y-axis)
Equation of the conjugate axis	x = 0 (y-axis)	y = 0 (x-axis)
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equations of the latera recta	x = ±ae	y = ±be

ILLUSTRATIVE EXAMPLES:

Example 1: Find the coordinates of the foci and the length of the latus rectum of the hyperbola $21x^2 - 4y^2 = 84$.

Solution: The equation of the hyperbola is,

$21x^2 - 4y^2 = 84$

i.e., $\frac{21x^2}{84}-\frac{4y^2}{84}=1$

i.e., $\frac{x^2}{4}-\frac{y^2}{21}=1$, Which is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Here, $a^2 = 4$, $b^2 = 21$

We have, $b^2 = a^2(e^2 - 1)$

$\Rightarrow 21=4\left(e^2-1\right)$

$\Rightarrow e^2-1=\frac{21}{4}$

$\Rightarrow e^2=1+\frac{21}{4}$

$$\Rightarrow e^2 = \frac{25}{4}$$

$$\Rightarrow e = \frac{5}{2}$$

$\therefore$ The coordinates of the foci are $(\pm ae, 0)$, i.e., $\left(\pm 2 \times \frac{5}{2}, 0\right)$,

i.e., $(\pm 5, 0)$ and the length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \times 21}{2} = 21$.

Example 2: Find the equation of the hyperbola referred to its axes as coordinate axes and its conjugate axis is 5 and the distance between its foci is 13.

Solution: Let the equation of the hyperbola referred to its axes as coordinate axes be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

According to the given conditon, we have 2b = 5,

i.e., b = $\frac{5}{2}$ and 2ae = 13, i.e., ae = $\frac{13}{2}$.

Thus using the relation $b^2 = a^2(e^2 - 1) = (ae)^2 - a^2$

We get, $\left(\frac{5}{2}\right)^2 = \left(\frac{13}{2}\right)^2 - a^2$

$\Rightarrow a^2 = \frac{169}{4} - \frac{25}{4} = \frac{144}{4} = 36$.

Hence the required equation of the hyperbola is, $\frac{x^2}{36} - \frac{y^2}{\left(\frac{5}{2}\right)^2} = 1$

i.e., $25x^2 - 144y^2 = 900$.

Example 3: If e_1 and e_2 be the eccentricities of a hyperbola and its conjugate, show that $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$.

Solution: Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be a given hyperbola with eccentricity e_1;

We have, $e_1 = \frac{\sqrt{a^2 + b^2}}{a}$.

Its conjugate hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ has eccentricity e_2;

We have, $e_2 = \frac{\sqrt{b^2 + a^2}}{b}$.

Hence, $\frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1$.



CHECK YOUR PROGRESS

Q.6: Find the eccentricity, length of the axes, coordinates of the vertices and of the foci, length of each latus rectum and the equation of the directrices of each of the following hyperbolas:

i) $9x^2 - 16y^2 = 144$ ii) $2x^2 - 3y^2 + 30 = 0$

Q.7: Find the standard equation of the hyperbola whose transverse axis is 12 and eccentricity is 4.

Q.8: Find the standard equation of a rectangular hyperbola whose latus rectum is of length 10 units.

5.6 GENERAL EQUATION OF THE SECOND DEGREE

The most general equation of the second degree in x and y is,

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ...(1)

such an equation always represents a conic (including improper conics viz.a circle and a pair of straight lines). It is useful to remember the following results,

Where, $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$.

The equation (1) represents

i) a parabola if, $ab - h^2 = 0$, provided $D \neq 0$;

ii) an ellipse if, $ab - h^2 > 0$, provided $D \neq 0$;

iii) a hyperbola if, $ab - h^2 < 0$, provided $D \neq 0$;

iv) a circle if, a = b and h = 0;

v) a pair of straight lines if, D = 0.

In case of (iii) if, a + b = 0 in addition to $ab - h^2 < 0$, (1) represents a rectangular hyperbola.

Example 1: What locus is represented by the equation $7x^2 - 2xy + 7y^2 - 6x + 16y - 7 = 0$?

Solution: It is a 2nd degree equation in x and y is; in usual notations,

We have, $abc + 2fgh - af^2 - bg^2 - ch^2$

$= 7.7.(-7) + 2.8.(-3).(-1) - 7.8^2 - 7.(-3)^2 - (-7).(-1)^2$

$= -243 + 48 - 448 - 63 + 7 \neq 0$

and $ab - h^2 = 7.7 - (-1)^2 = 49 - 1 = 48 > 0$.

$\therefore$ The given equation represents an ellipse.

Example 2: Find the value of h, so that the equation $25x^2 + 2hxy + 9y^2 + 3x - 7y + 13 = 0$ may represent a parabola.

Solution : In order that the given equation $25x^2 + 2hxy + 9y^2 + 3x - 7y + 13 = 0$ may represent a parabola, we must have,

$25.9 - h^2 = 0$.

$\Rightarrow h^2 = 25.9 = (5 \times 3)^2$

$\Rightarrow h = \pm 15$



CHECK YOUR PROGRESS

Q.9: What locus is represented by the equation $3x^2 - 8xy + 4y^2 - 3x + 5y + 11 = 0$?

Q.10: Find the value of λ so that the equation $9x^2 + 21xy + \lambda y^2 - 3x + 7y - 17 = 0$ may represent a parabola.



5.7 LET US SUM UP

- A conic, whose eccentricity is 1, is called a parabola. The standard equation of a parabola is $y^2 = 4ax$.

- A conic, whose eccentricity is less than 1, is called an ellipse. The standard of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
- A conic, whose eccentricity is greater than 1, is called a hyperbola. The standard of aa hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.
- The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents
  - i) a parabola if, $ab - h^2 = 0$, provided $D \neq 0$;
  - ii) an ellipse if, $ab - h^2 > 0$, provided $D \neq 0$;
  - iii) a hyperbola if, $ab - h^2 < 0$, provided $D \neq 0$;
  - iv) a circle if, $a = b$ and $h = 0$;
  - v) a pair of straight lines if, $D = 0$, Where, $D = abc + 2fgh - af^2 - bg^2 - ch^2$

  In case of (iii) if, $a + b = 0$ in addition to $ab - h^2 < 0$, (1) represents a rectangular hyperbola.



5.8 FURTHER READING

1) Jain, P. K. and Ahmhad, K. *Textbook of Analytical Geometry of Three Dimensions*. New Delhi: New Age International (P) Ltd.
2) Kishan, Hari. *Coordinate Geometry of Two dimensions*. New Delhi: Atlantic Publishers & Distributors.
3) Pal, Sat and Pal, Harbans. *A Textbook of Two Dimensional Geometry*. New Age International Publishers.
4) Sharma, G. C. and Jain, Madhu. *Coordinate Geometry (2D and 3D)*. New Delhi: Galgotia Publications Pvt Ltd.
5) Sharma, A. K. *Coordinate Geometry (2D)*. New Delhi: Discovery Publishing House.



5.9 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: i) The equation of the parabola is, $3y^2 = 8x$, i.e., $y^2 = \frac{8}{3}x$

Which is of the form $y^2 = 4ax$, where, $4a = \frac{8}{3}$, i.e., $a = \frac{2}{3}$.

For the parabola, we have the coordinates of the focus are (a, 0), i.e., $\left(\frac{2}{3}, 0\right)$; the coordinates of the vertex are (0, 0); the equation of the directrix is x = –a, i.e., $x = -\frac{2}{3}$, i.e., 3x + 2 = 0; the equation of the axis of the parabola is y = 0; the length of the latus rectum $= 4a = \frac{8}{3}$.

ii) The parabola is given by $y^2 = –8x$.
Which is of the form $y^2 = –4ax$ (a > 0), where, 4a = 8, i.e., a = 2.
Hence, the coordinates of the focus are (–a, 0), i.e., (–2, 0); the vertex has coordinates (0, 0); the equation of the directrix is x = a, i.e., x = 2; the equation of the axis is y = 0; the length of the latus rectum = 4a = 8.

iii) The equation of the parabola is $x^2 = 10y$, which is of the form $x^2 = 4ay$ (a > 0), where, 4a = 10, i.e., $a = \frac{5}{2}$.
For the parabola the focus has coordinates (0, a), i.e., $\left(0, \frac{5}{2}\right)$; the vertex has coordinates (0, 0); the directrix has equation y = – a, i.e., $y = -\frac{5}{2}$, i.e., 2y + 5 = 0; the axis has equation x = 0; the length of the latus rectum = 4a = 10.

iv) The equation of the parabola is $x^2 = –8y$, which is of the form $x^2 = –4ay$ (a > 0); here, 4a = 8, i.e., a = 2.
It focus has coordinates (0, –a), i.e., (0, –2); its vertex is at (0, 0); its directrix has equation y = a, i.e., y = 2; its axis has equation x = 0; its latus rectum has length 4a i.e. 8.

v) The given equation is $x^2 – 6x + 4y + 1 = 0$ ... ...(1)

i.e., $x^2 – 6x = –4y – 1$

i.e., $x^2 – 6x + 9 = –4y – 1 + 9$

i.e., $(x – 3)^2 = –4y + 8$

i.e., $(x – 3)^2 = –4(y – 2)$

Transferring the origin to the point (3, 2), without changing directions of the coordinate axes, the above equation becomes $X^2 = -4Y$

Hence the given equation (1) represents a parabola whose equation referred to the parallel axes through the new origin (3, 2), is $X^2 = -4Y$, which is of the form $X^2 = -4aY$ $(a > 0)$, where $a = 1$; referred to the new axes, the parabola has its vertex at (0, 0), focus at (0, –a), i.e., (0, –1), its axis the new Y-axis having equation $X = 0$ and its directrix has equation $Y = a$, i.e., $Y = 1$.

Since $x = X + 3$ and $y = Y + 2$, we see that,

$(X, Y) = (0, 0) \Leftrightarrow (x, y) = (3, 2)$,

$(X, Y) = (0, -1) \Leftrightarrow (x, y) = (3, 1)$,

$X = 0 \Leftrightarrow x = 3$,

$Y = 1 \Leftrightarrow y = 3$.

Hence referred to the original axes the parabola has its vertex at (3, 2), focus at (3, 1), has the line $x = 3$ (parallel to y-axis) as its axis and the line $y = 3$ (parallel to the x-axis) as its directrix.

Also the length of the latus rectum of the parabola $= 4a = 4$.

Ans. to Q. No. 2: We know that the equation of the parabola with focus (a, 0) and directrix $x = -a$ is $y^2 = 4ax$. Hence the equation of the parabola with focus (4, 0) and directrix $x = -4$ (for which $a = 4$) is $y^2 = 4.4x$, i.e., $y^2 = 16x$.

Ans. to Q. No. 3: The required parabola has its vertex at the origin and focus on the positive side of the y-axis.Let the equation of the parabola be $x^2 = 4ay$. As the length of its latus rectum is 20, we have $4a = 20$ giving $a = 5$.

$\therefore$ The required equation of the parabola is $x^2 = 20y$.

Ans. to Q. No. 4: i) The ellipse has equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$, i.e., $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$

which is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Here, $a = 5$, $b = 3$, so that $a > b$. So the major axis lies on x-axis.

We have $e = \frac{\sqrt{5^2 - 3^2}}{5} = \frac{4}{5}$.

For the given ellipse, we have,

the length of the major axis = 2a = 10;

the length of the minor axis = 2b = 6;

the eccentricity = e = $\frac{4}{5}$;

the coordinates of the vertices are (±a, 0), i.e., (±5, 0);

the coordinates of the foci are (±ae, 0), i.e., (±4, 0);

the length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \times 3^2}{5} = \frac{18}{5}$;

the equations of the directrices are,

x = $\pm \frac{a}{e}$, i.e., x = $\pm \frac{5}{\frac{4}{5}}$, i.e., x = $\pm \frac{25}{4}$.

ii) The equation of the ellipse is, $3x^2 + 2y^2 = 18$

$$\text{i.e., } \frac{3x^2}{18} + \frac{2y^2}{18} = 1$$

$$\text{i.e., } \frac{x^2}{\left(\sqrt{6}\right)^2} + \frac{y^2}{3^2} = 1$$

Which is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Here, a = $\sqrt{6}$ and b = 3;

Clearly b > a and so the major axis of length = 2b = 6, lies on the y-axis.

In this case e = $\frac{\sqrt{b^2 - a^2}}{b} = \frac{\sqrt{9-6}}{3} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$.

For the above ellipse, we have,

the length of the major axis = 2b = 6;

the length of the minor axis = 2a = $2\sqrt{6}$;

the eccentricity = e = $\frac{1}{\sqrt{3}}$;

the coordinates of the vertices are (0, ±b), i.e., (0, ±3);

the cordinates of the foci are (0, ±be), i.e., (0, ± $\sqrt{3}$);

the equations of the directrices are,

$y = \pm\frac{b}{e}$, i.e., $y = \pm\frac{3}{\frac{1}{\sqrt{3}}}$, i.e., $y = \pm 3\sqrt{3}$;

the length of the latus rectum = $\frac{2a^2}{b} = \frac{2\times 6}{3}$ = 4.

Ans. to Q. No. 5: As the centre of the ellipse is at the origin and its major axis lies on the x-axis, its equation is of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{... ... ...(1)}$$

Where a and b are the lengths of its semi-major axis and semi-minor axis respectively. Here, as given, eccentricity e = $\frac{2}{3}$ and the length of the latus rectum = $\frac{2b^2}{a}$ = 5.

We get, $2b^2 = 5a$

$\Rightarrow 2.a^2(1-e^2) = 5a$

$\Rightarrow 2a\left\{1-\left(\frac{2}{3}\right)^2\right\} = 5$

$\Rightarrow 2a\times\frac{5}{9}$

$\Rightarrow a = \frac{9}{2}$

and hence b = $\frac{5a}{2} = \frac{5}{2}\times\frac{9}{2} = \frac{45}{4}$.

Therefore (1) becomes $\frac{x^2}{\frac{81}{4}}+\frac{y^2}{\frac{45}{4}}=1$, i.e., $\frac{4x^2}{81}+\frac{4y^2}{45}=1$

Which is the required equation of the ellipse.This can be expressed also as $20x^2 + 36y^2 = 405$.

Ans. to Q. No. 6: i) The equation of the hyperbola is, $9x^2 - 16y^2 = 144$

i.e., $\frac{x^2}{16}-\frac{y^2}{9}=1$

i.e., $\frac{x^2}{4^2}-\frac{y^2}{3^2}=1$

Which is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with a = 4, b = 3.

Its transverse axis lies on the x-axis.

Now, e = $\frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{4^2 + 3^2}}{4} = \frac{5}{4}$.

Thus, eccentricity of the hyperbola = e = $\frac{5}{4}$;

length of transverse axis = 2a = 8;

length of conjugate axis = 2b = 6;

coordinates of the vertices are (±a, 0), i.e., (±4, 0);

coordinates of the foci are (±ae, 0), i.e., (±5, 0);

length of each latus rectum = $\frac{2b^2}{a} = \frac{2 \times 3^2}{4} = \frac{9}{2}$;

the equations of the directrices are x = $\pm \frac{a}{e}$

i.e., x = $\pm \frac{4}{\frac{5}{4}}$, i.e., x = $\pm \frac{16}{5}$.

ii) The equation of the hyperbola is, $2x^2 - 3y^2 + 30 = 0$

i.e., $3y^2 - 2x^2 = 30$

i.e., $\frac{3y^2}{30} - \frac{2x^2}{30} = 1$

i.e., $\frac{y^2}{\left(\sqrt{10}\right)^2} - \frac{x^2}{\left(\sqrt{15}\right)^2} = 1$

Which is of the form $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ with b = $\sqrt{10}$ and a = $\sqrt{15}$; its

transverse axis lies on the y-axis.

We have, e = $\frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{15 + 10}}{\sqrt{10}} = \frac{5}{\sqrt{10}} = \sqrt{\frac{5}{2}}$.

For this hyperbola we have,

the eccentricity = e = $\sqrt{\frac{5}{2}}$;

the length of transverse axis = 2b = $2\sqrt{10}$;

the length of conjugate axis = 2a = $2\sqrt{15}$;

the coordinates of the vertices are (0, ±b), i.e., (0, $\pm\sqrt{10}$);

the coordinates of the foci are (0, ±be), i.e., (0, ±5);

the length of each latus rectum = $\frac{2a^2}{b} = \frac{2\times 15}{\sqrt{10}} = 3\sqrt{10}$;

the equations of the directrices are, y = $\pm\frac{b}{e}$

i.e., y = $\pm\frac{\sqrt{10}}{\sqrt{\frac{5}{2}}}$,

i.e., y = ±2.

Ans. to Q. No. 7: Let the standard equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

By the given conditions, 2a = 12 and e = 4 $\Rightarrow \alpha$ = 6 and e = 4.

Then, $b^2 = a^2(e^2 - 1) = 6^2(4^2 - 1) = 36 \times 15 = 540$

So the required standard equation of the hyperbola is $\frac{x^2}{36} - \frac{y^2}{540} = 1$.

Ans. to Q. No. 8: Let the standard equation of the rectangular hyperbola be $x^2 - y^2 = a^2$. As its latus rectum has length 10 units,

We get, $\frac{2a^2}{a}$ = 10

$\Rightarrow$ 2a = 10

$\Rightarrow \alpha$ = 5.

So the required equation of the rectangular hyperbola is $x^2 - y^2 = 25$.

Ans. to Q. No. 9: Given, $3x^2 - 8xy + 4y^2 - 3x + 5y + 11 = 0$

It is a 2nd degree equation in x & y. In usual symbols,

we have, $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 3.4.11 + 2.\frac{5}{2}.\left(-\frac{3}{2}\right)(-4) - 3.\left(\frac{5}{2}\right)^2 - 4.\left(-\frac{3}{2}\right)^2 - 11.(-4)^2$$

$$= 132 + 30 - \frac{75}{4} - 9 - 176$$

$$= -\frac{75}{4} - 23 \neq 0$$

and $ab - h^2 = 3.4 - (-4)^2 = 12 - 16 = -4 < 0$

So, the given equation represents a hyperbola.

Ans. to Q. No. 10: In order that $9x^2 + 21xy + \lambda y^2 - 3x + 7y - 17 = 0$ may represent a parabola,

we must have, $9.\lambda - \left(\frac{21}{2}\right)^2 = 0$ (condition: $ab - h^2 = 0$)

$$\Rightarrow 9\lambda = \frac{21 \times 21}{4}$$

$$\Rightarrow \lambda = \frac{21 \times 21}{4 \times 9}$$

$$\Rightarrow \lambda = \frac{49}{4}.$$



5.10 MODEL QUESTIONS

Q.1: Find the vertex, focus, equation of the axis and directrix of the parabola $(y + 3)^2 = 2(x + 2)$.

Q.2: Find the coordinates of the focus and the vertex, the equation of the axis, directrix and the latus rectum for the parabola,
$x^2 + 4x + 4y + 16 = 0$.

Q.3: Find the coordinates of the focus and the vertex, the equation of the axis and directrix of the parabola, $y^2 - 4y - 2x - 8 = 0$.

Q.4: Find the coordinates of the focus and the vertex, the equation of the axis, directrix and latus rectum of the parabola,

i) $4y^2 - 6x - 4y = 5$ ii) $x^2 - y - 2x = 0$ iii) $x^2 + 2y = 8x - 7$.

Q.5: Find the eccentricity, foci and latus rectum of the ellipse,

i) $2x^2 + 3y^2 = 6$ ii) $3x^2 + 2y^2 = 6$

Q.6: Find the equation of the ellipse referred to its axes as the axes of the co-ordinates whose foci are (±2, 0) and eccentricity is $\frac{1}{2}$.

Q.7: Find the eccentricity of the ellipse, (i) if its major axis is twice the minor axis, (ii) if its latus rectum is half of its minor axis.

Q.8: Find the equation of the hyperbola with foci (±2, 0) and eccentricity $\frac{3}{2}$.

Q.9: Find the equation of the hyperbola having distance between directrices $\frac{4\sqrt{3}}{3}$ and passng through the point (2, 1).

Q.10: Find the equation of the hyperbola whose directrix is 2x + y = 1, focus (1, 2) and eccentricity is $\sqrt{3}$.

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UNIT 6: INTRODUCTION OF COORDINATE IN THREE DIMENSION

UNIT STUCTURE

6.1 Learning Objectives

6.2 Introduction

6.3 Rectangular Catesian Coordinates in Three Dimensions

6.4 Change of Origin (Without Changing the Direction of Axes)

6.5 Distance Between Two Points

6.6 Section Formula

6.7 Direction Cosines of a Line

6.8 Direction Ratios of a Line

6.9 Let Us Sum Up

6.10 Further Reading

6.11 Answers to Check Your Progress

6.12 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- explain coordinate axes in three dimensions
- find distance between two points and division of the join of two points in a given ratio
- know direction cosine and direction ratios of a line.

6.2 INTRODUCTION

We have already discussed about two dimensinal geometry. In this unit, we will discuss in details about the co-ordinate system and co-ordinates of a point in space, distance between two points in space, position of a point dividing the join of two points in a given ratio internally/externally. We will also discuss equation of plane and line in space in this unit.

6.3 RECTANGULAR CARTESIAN COORDINATES IN THREE DIMENSIONS

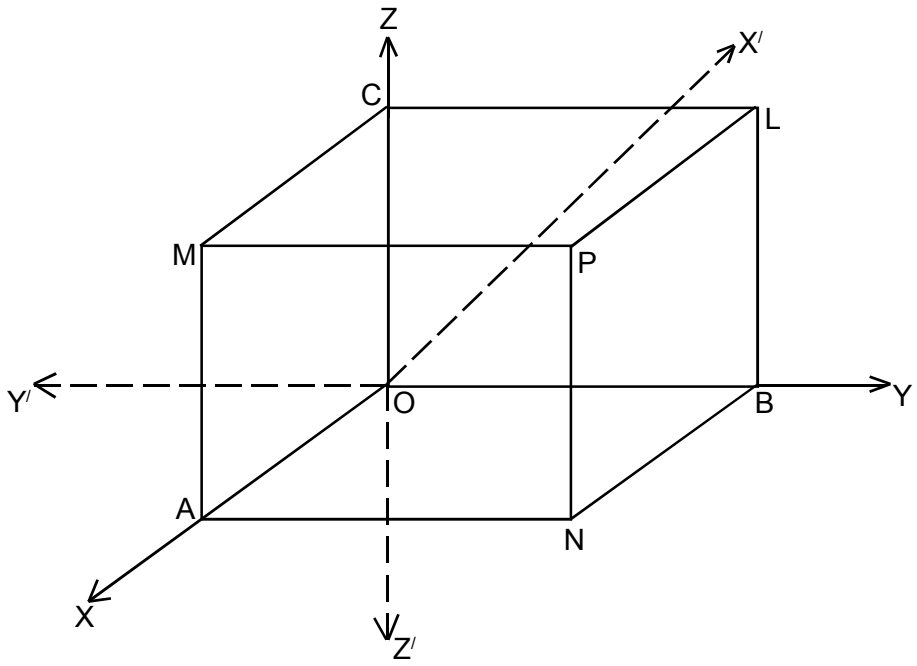


Fig. 6.1

Through any fixed point O in space (our familiar space of three dimensions) we take three mutually perpendicular straight lines $X'OX$, $Y'OY$ and $Z'OZ$ (as shown in the above figure) which are called the x-axis, y-axis and z-axis respectively, O being called the origin. Conventially, the direction OX, OY, OZ are chosen as positive direction of the respective axes while the direction OX', OY', OZ' are their negative direction. These three mutually perpendicular axes, called the coordinate axes, give rise to the three mutually perpendicular planes YOZ, ZOX, XOY known as the coordinate planes. The YOZ plane,which contains the y and z axes,is called the yz-plane (or the x-plane, the x-axis being perpendicular to it), the ZOX plane is called the zx-plane (or the y-plane) and the XOY plane is called the xy-plane (or the z-plane).

Let P be any point in space.Through P, we draw three planes parallel respectively to the coordinate planes YOZ, ZOX, XOY cutting the coordinate axes XX', YY', ZZ' at A, B, C respectively. These three planes along with the three coordinate planes form a rectangular parallelopiped of which OA,

OB, OC are three coterminous edges and OP is a diagonal. Then the position of P will be known when the directed line segments OA, OB, OC (or their equivalent segments) are given in magnitude and sign.

Let OA = LP = BN = CM = x,

OB = MP = AN = CL = y,

OC = NP = AM = BL = z

Here x, y, z are real numbers which may be positive,negative or zero. x is positive if A lies on the positive side OX of the x-axis; x is negative if A lies on its negative side OX'; x is zero if A coincides with the origin O which will be the case if P lies on the yz-plane so that the parallelopiped with diagonal OP degenerates into a rectangle in the yz-plane with OP as a diagonal. Note that the length of the segment OA (or their equivalents) is |x|. Similar is the case with the real numbers y and z. To each point P in space there correponds a unique triplet (x, y, z) of real numbers and vice-versa. We say that the point P has rectangular cartesian coordinates (x, y, z) referred to the coordinate axes $X'OX$, $Y'OY$ and $Z'OZ$. We usually write P(x, y, z) to mean that the point P has coordinates (x, y, z) and often say' P is the point (x, y, z)' to mean the same thing. Also,we say that (x, y, z) is a point to mean that (x, y, z) are the coordinates of a point.Clearly for the point O all the three points L, M, N coincide with O so that x = 0, y = 0, z = 0; hence the coordinates of the origin O are (0, 0, 0). Referring to figure 6.1, we notice that PL, PM, PN are the perpendiculars drawn from the point P to the coordinate plane YOZ, ZOX, XOY respectively. As such the coordinates x, y, z of P are the perpendicular distances of P from the yz-plane, the zx-plane,the xy-plane respectively with proper signs. Further $\angle OCP = 90^0$; so z = OC is the (orthogonal)projection of OP on z-axis. Likewise x = OA is the projection of OP on x-axis and y = OB is the projection of OP on y-axis.

The three coordinate planes divide the entire space into eight parts each of which is called an octant and the signs of the coordinates of a point determine the octant in which it lies. The following table gives the signs of the coordinates in different octants:

OCTANT / COORDINATE	I OXYZ	II $OX'YZ$	III $OX'Y'Z$	IV $OXY'Z$	V $OXYZ'$	VI $OX'YZ'$	VII $OX'Y'Z'$	VIII $OXY'Z'$
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

The system described above which enables us to locate the position of any point in space relative to three mutually perpendicular axes is called a rectangular cartesian coordinate system.In figure 6.1, the postive directions OX, OY, OZ of the coordinate axes have been chosen in such a manner that if a right-handed corkscrew is turned from direction OX to direction OY, then the screwhead moves in the direction OZ. The system as shown in figure 6.1 is called a right-handed coordinate system.

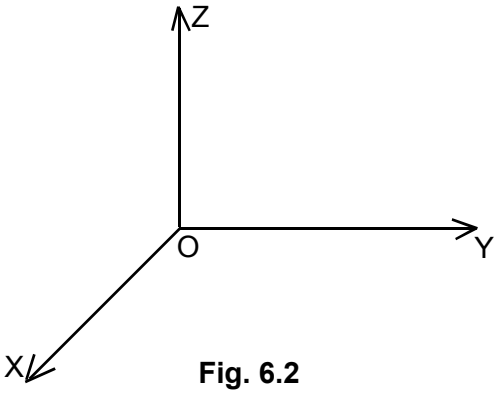


Fig. 6.2

On the other hand if the positive direction are chosen as in the adjoining figure 6.2, the coordinate system is said to be left-handed. For uniformity we shall use right-handed system.

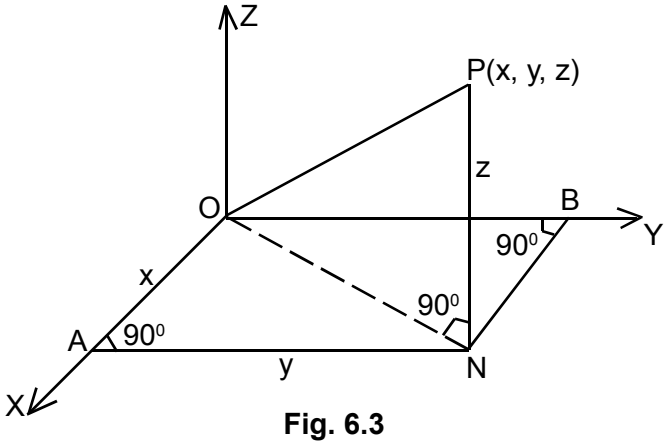


Fig. 6.3

Let P(x, y, z) be any point in space and let PN be drawn perpendicular from P to the xy-plane,NA perpendicular to x-axis and NB perpendicular to y-axis as shown in the diagram above. Then we see that OA = BN = x, OB = AN = y and NP = z.This simplified diagramatic representation of the coordinates is quite convenient. We note that PN is perpendicular to every line in the xy-plane and so $\angle$ONP is a right angle.

From the right-angled triangle OPN,

We have, $OP^2 = ON^2 + NP^2$ and from the right-angled triangle OAN,

We have, $ON^2 = OA^2 + AN^2$

Hence, $OP^2 = OA^2 + AN^2 + NP^2 = x^2 + y^2 + z^2$

So that, OP = $\sqrt{x^2 + y^2 + z^2}$

Therefore the distance between the origin and any point P(x, y, z) is, $\sqrt{x^2 + y^2 + z^2}$.

6.4 CHANGE OF ORIGIN (WITHOUT CHANGING THE DIRECTION OF AXES)

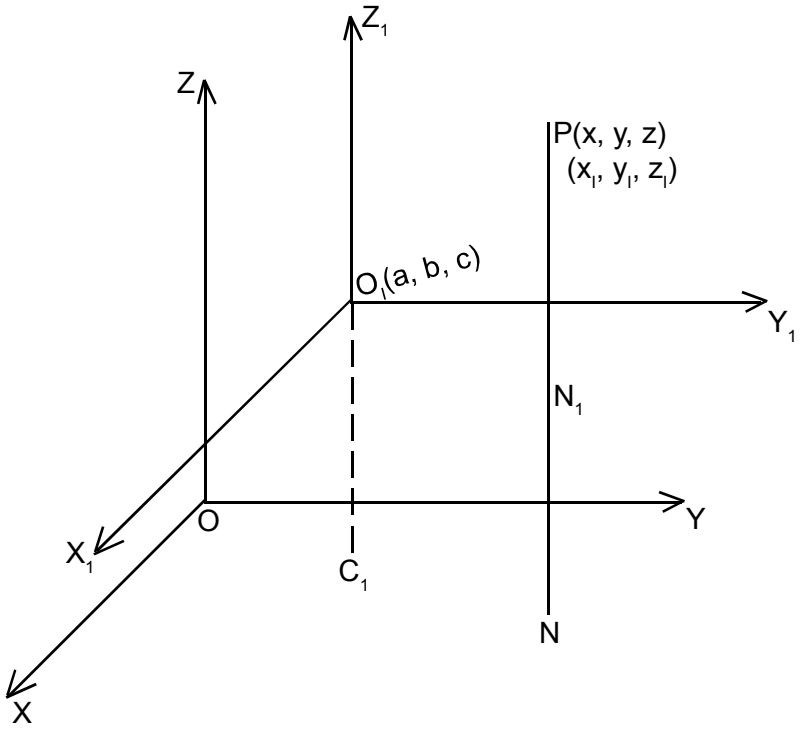


Fig. 6.4

Let (a, b, c) be the coordinates of a point O_1 referred to the coordinate axes OX, OY, OZ. Through the point O_1 take new coordinate axes O_1X_1, O_1Y_1, O_1Z_1 parallel repectively to the old axes OX, OY, OZ in the same direction. Let P be any point in space having coordinates (x, y, z) referred to the old set and (x_1, y_1, z_1) referred to the new set of axes. Draw a line through P parallel to OZ (and hence parallel to O_1Z_1) which will be perpendicular to both XOY and $X_1O_1Y_1$ planes.Suppose it intersects these planes at N and N_1 respectively.

Then, $NP = z$, $N_1P = z_1$

C_1O_1 = z-coordinate of the point $O_1(a, b, c)$ referred to the old axes = c.

So, $z = NP = NN_1 + N_1P = C_1O_1 + N_1P = c + z_1$

Hence, $z_1 = z - c$

Drawing perpendicular to the other coordinate planes, we similarly get, $x_1 = x - a$, and $y_1 = y - b$

Thus, $x_1 = x - a$, $y_1 = y - b$ and $z_1 = z - c$ are the new coordinates of the point P(x, y, z) referred to the point $O_1(a, b, c)$ as the origin.

6.5 DISTANCE BETWEEN TWO POINTS

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two given points in space. If we transform the origin to the point P (without changing the directions of the axes),then the new coordinates of the point Q are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. Thus referred to the new set of axes P is the origin and Q has coordinates $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. so that,on using the result obtained at the end of 6.3 above, we have,

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

6.6 SECTION FORMULA

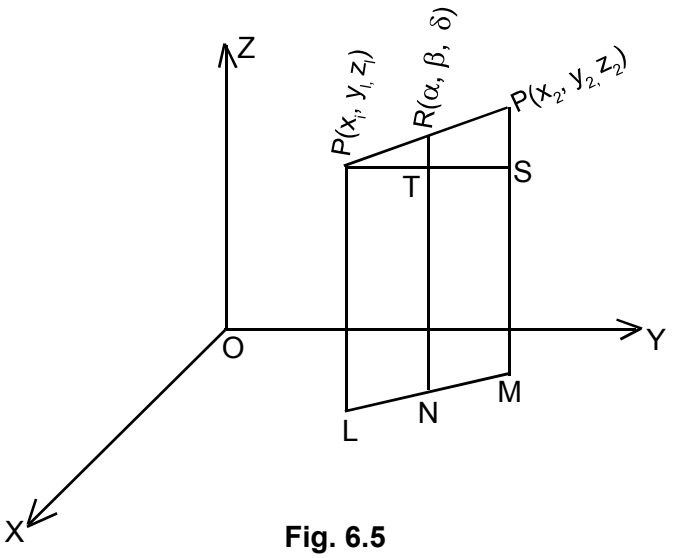


Fig. 6.5

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two given points.Suppose that the point R divides the join of P and Q internally in the ratio m : n as shown in figure 6.5. We have to find the coordinates (α, β, γ) of R.

Let PL, QM, RN be the perpendiculars drawn on the xy-plane; then L, M, N lie in a straight line in the xy-plane as shown.Also through P draw a line parallel to the line LM which intersects QM and RN at right angles at S and T repectively.From similar triangles PRT and PQS, we have,

$$\frac{PR}{PQ} = \frac{TR}{SQ}$$

also given, $\frac{PR}{RQ} = \frac{m}{n}$

$$\Rightarrow \frac{PR}{PR + RQ} = \frac{m}{m+n}$$

$$\Rightarrow \frac{PR}{PQ} = \frac{m}{m+n}$$

So we get, $\frac{TR}{SQ} = \frac{m}{m+n}$

$$\Rightarrow \frac{NR - NT}{MQ - MS} = \frac{m}{m+n}$$

$$\Rightarrow \frac{NR - LP}{MQ - LP} = \frac{m}{m+n}$$

$$\Rightarrow \frac{\gamma - z_1}{z_2 - z_1} = \frac{m}{m+n}$$

$$\Rightarrow (\gamma - z_1)(m + n) = m(z_2 - z_1)$$

$$\Rightarrow (m + n)\gamma = m(z_2 - z_1) + (m + n)z_1$$

$$\Rightarrow (m + n)\gamma = (m + n)\gamma = mz_2 + nz_1$$

$$\Rightarrow \gamma = \frac{mz_2 + nz_1}{m+n}$$

Similarly drawing perpendiculars to the yz-plane and zx-plane,

we get, $\alpha = \frac{mx_2 + nx_1}{m+n}$, $\beta = \frac{my_2 + ny_1}{m+n}$

Hence the coordinates of R are,

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

If R divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio m : n,

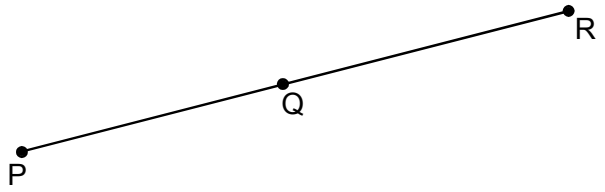


Fig. 6.6

then it can easily be seen that the coordinates of R are,

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

The above two results can be combined as follows:

If the point R divides the join of the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio λ : 1 ($\lambda \neq 1$), then the coordinates of R are,

$$\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1}\right)$$

If λ is positive, R lies between P and Q and the division is internal; if λ is negative, R lies on the same side of P and Q, the division being external.

Note:

1) The midpoint of PQ is, $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2}\right)$

2) The centroid of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is,

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Remarks:

1) For every point on the x-axis, y = 0, z = 0; for every point on the y-axis, z = 0, x = 0; for every point on the z-axis, x = 0, y = 0. Thus an arbitrary point on the x-axis has coordinates of the form (x, 0, 0) and so on.
2) For any point on the yz-plane x = 0; for any point on the zx-plane y = 0 and for any point on the xy-plane z = 0.
3) We have taken the three coordinate axes to be mutually perpendicular. It is however possible to locate the position of a point in space with reference to any three non-coplanar lines through a fixed point (origin) as the coordinate axes. In this case the axes are oblique. We shall have no occasion to use oblique axes in our discussion.

The Equation to a Surface: Consider the equation x = 0. It is satisfied by the coordinates of every point on the yz-plane(and by the coordinates of no point outside it). We say that the equation x = 0 represents the yz-plane, or equivalently,the equation of the yz-plane is x = 0. Likewise y = 0 represents the zx-plane and z = 0 represents the xy-plane.

Thus the equations of the three coordinate planes are x = 0, y = 0 and z = 0.

Next consider the equation $x^2 + y^2 + z^2 = a^2$. As we have already seen that $\sqrt{x^2+y^2+z^2}$ is the distance of the point (x, y, z) from the origin, the above equation is true for every point at the same distance 'a' from the origin. But the locus of all such points is the sphere whose centre is at the origin and whose radius is 'a'.

Hence the equation $x^2 + y^2 + z^2 = a^2$ represents the sphere whose centre is at the origin and whose radius is 'a'.

In general any equation involving one or more of the current coordinates (x, y, z) of a variable point represents a surface or a system of surfaces which is the locus of the variable point.

The Equations to a Curve in Space: Two equations f(x, y, z) = 0 and g(x, y, z) = 0 taken separately represent two surfaces. They are simultaneously satisfied by the coordinates of the points which are common to the two surfaces. These common points constitute a curve in space which is the curve of intersection of the two surfaces. Thus the simultaneous equations f(x, y, z) = 0 and g(x, y, z) = 0 represent a curve in space, which is the curve in which the two surfaces f = 0 and g = 0 intersect.

ILLUSTRATIVE EXAMPLES:

Example 1: State in which octant the point (–2, –1, 7) lies.

Solution: For the point (–2, –1, 7), x = –2 < 0, y = –1 < 0, z = 7 > 0.

$\therefore$ The (–2, –1, 7) lies in octant $OX'Y'Z$ (octant III).

Example 2: Given the points A(3, 4, 5) and B(–1, 3, –7), find the equation of the locus of moving point P which is always equidistant from A and B.

Solution: Let (x, y, z) be the coordinates of the moving point P. As P(x, y, z) is always equidistant from A(3, 4, 5) and B(–1, 3, –7), we have,

$PA = PB$

$\Rightarrow PA^2 = PB^2$

$\Rightarrow (x-3)^2 + (y-4)^2 + (z-5)^2 = (x+1)^2 + (y-3)^2 + (z+7)^2$

$\Rightarrow (x^2 - 6x + 9) + (y^2 - 8y + 16) + (z^2 - 10z + 25) =$

$(x^2 + 2x + 1) + (y^2 - 6y + 9) + (z^2 + 14z + 49)$

$\Rightarrow -8x - 2y - 24z - 9 = 0$

$\Rightarrow 8x + 2y + 24z + 9 = 0$

This is the required equation of the locus of P.

Example 3: Find the point where the line joining the points A(–2, 4, 7) and B(3, –5, 8) meet the xy-plane.

Solution: Suppose that the line joining the points A(–2, 4, 7) and B(3, –5, 8) meet the xy-plane at the point P. As P lies in the xy-plane, its z coordinate is zero; let the coordinates of P be $(\alpha, \beta, 0)$.

Let AP : PB = λ : 1; then we must have,

$$\frac{\lambda.8 + 1.7}{\lambda + 1} = 0 \Rightarrow 8\lambda + 7 = 0 \Rightarrow 8\lambda = -7 \Rightarrow \lambda = -\frac{7}{8}$$

Hence, $\alpha = \frac{\lambda.3+1.(-2)}{\lambda+1}$

$= \frac{\left(-\frac{7}{8}\right).3-2}{-\frac{7}{8}+1}$

$= \frac{-\frac{37}{8}}{\frac{1}{8}}$

$= -37$

and $\beta = \frac{\lambda.(-5)+1.4}{\lambda+1}$

$= \frac{\left(-\frac{7}{8}\right).(-5)+4}{-\frac{7}{8}+1}$

$= \frac{\frac{67}{8}}{\frac{1}{8}}$

$= 67$

$\therefore$ P has coordinate (–37, 67, 0)

Thus the line joining the given points A and B meet the xy plane at P (–37, 67, 0).

Example 4: A(1, 2, 3), B(–1, –2, –1), C(2, 3, 2) and D are the vertices of a paralleogram. Find the coordinates of D. Also, show that ABCD is not a rectangle.

Solution: Suppose that the diagonals AC and BD meet at P; then P is the mid-point of each diagonal.

As P is the midpoint of AC, P has cordinates,

$\left(\frac{1+2}{2}, \frac{2+3}{2}, \frac{3+2}{2}\right)$, i.e., $\left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$.

Let (x, y, z) be the coordinates of D.

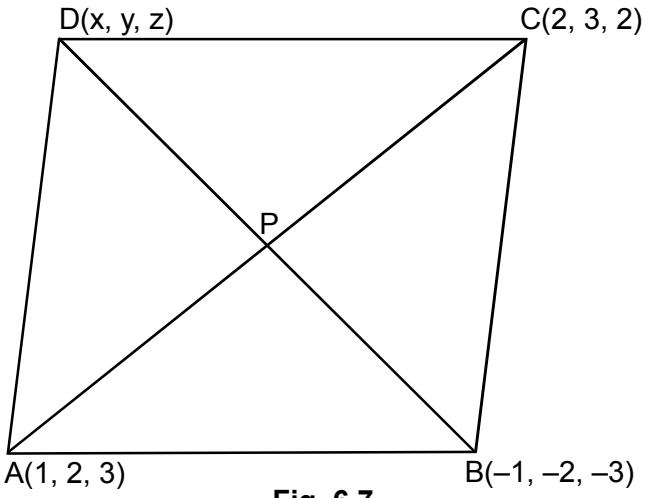


Fig. 6.7

Then, as $P\left(\frac{3}{2},\frac{5}{2},\frac{5}{2}\right)$ is mid point of BD joining B(–1, –2, –1) and D(x, y, z), we must have,

$$\frac{-1+x}{2}=\frac{3}{2},\ \frac{-2+y}{2}=\frac{5}{2},\ \frac{-1+z}{2}=\frac{5}{2}$$

$\Rightarrow$ x – 1 = 3, y – 2 = 5, z – 1 = 5

$\Rightarrow$ x = 4, y = 7, z = 6

Hence the coordinates of the vertex D are (4, 7, 6).

Now, AC = $\sqrt{(1-2)^2+(2-3)^2+(3-2)^2}$

= $\sqrt{1+1+1}$

= $\sqrt{3}$

and BD = $\sqrt{(4+1)^2+(7+2)^2+(6+1)^2}$

= $\sqrt{25+81+49}$

= $\sqrt{55}$

As AC $\neq$ BD, the paralleogram ABCD is not a rectangle.

Example 5: Find the point on the y-axis which is equidistant from the points (0, 3, –1) and (5, –4, 2).

Solution: Any point on the y-axis has coordinate of the form (0, y, 0). If it is equidistant from the points (0, 3, –1) and (5, –4, 2), then,

$$\sqrt{(0-0)^2+(y-3)^2+(0+1)^2}=\sqrt{(0-5)^2+(y+4)^2+(0-2)^2}$$

$\Rightarrow$ $(y – 3)^2 + 1 = 25 + (y + 4)^2 + 4$

$\Rightarrow$ $y^2 – 6y + 9 + 1 = y^2 + 8y + 16 + 29$

$\Rightarrow$ –6y – 8y = 45 – 10

$\Rightarrow$ –14y = 35

$\Rightarrow$ $y = -\frac{5}{2}$

$\therefore$ The required point on the y-axis is $\left(0, -\frac{5}{2}, 0\right)$.



CHECK YOUR PROGRESS

Q.1: State in which octants the following points lie

a) (–2, –1, 5) b) (1, 1, 7) c) (–1, –1, –2)

Q.2: Find the centre of the sphere which passes through the points O(0, 0, 0), A(a, 0, 0), B(0, b, 0) and C(0, 0, c). Also find its radius.

Q.3: Find the coordinates of the centoid G of the triangle whose vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

Q.4: Find the point where the line joining the point P(–2, 4, 7) and Q(3, –5, 8) meets the x-coordinate plane.

6.7 DIRECTION COSINES OF A LINE

If a line in space makes angles α, β, γ with the positive directions of x-axis, y-axis, z-axis repectively, then α, β, γ are called the direction angles of the line and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the line. Clearly all parallel lines have the same direction cosines.

As the x-axis makes angles 0^0, 90^0, 90^0 with the coordinate axes,the x-axis (or any line parallel to the x-axis) has direction cosines 1, 0, 0. Likewise the direction cosines of the y-axis (or its parallel) are 0,1,0 and the z-axis (or its parallel) are 0, 0, 1.

Relation between the direction cosines of a line:

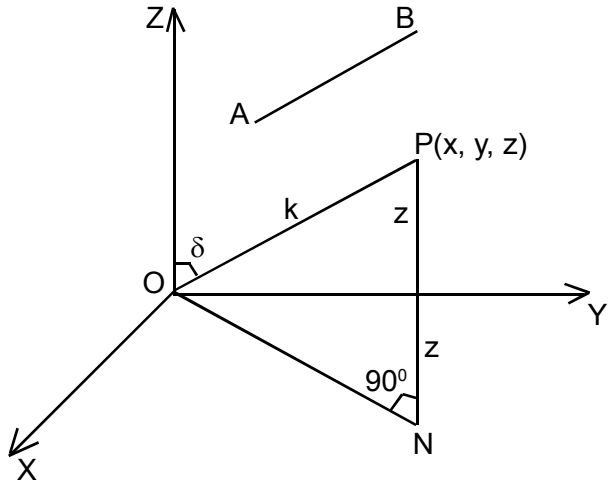


Fig. 6.8

Let AB be any line in space and suppose that it makes α, β, γ with OX, OY, OZ respectively. Through the origin O draw a line parallel to AB, and let P(x, y, z) be any point on this line. Then the line OP makes angles α, β, γ with OX, OY, OZ respectively. Let OP = k and let PN be the perpendicular from P to the xy-plane.

Then, NP = z and $\angle OPN = \angle ZOP = \gamma$.

From ΔOPN in which $\angle ONP = 90^0$,

We have, $NP = OP\cos\angle OPN \Rightarrow z = k\cos\gamma$

Similarly by drawing perpendiculars to the yz-plane and zx-plane, we get,

$x = k\cos\alpha$ and $y = k\cos\beta$

Again, $k^2 = OP^2 = x^2 + y^2 + z^2$

So, we get, $k^2 = (k\cos\alpha)^2 + (k\cos\beta)^2 + (k\cos\gamma)^2$

So that, $\cos^2\alpha + coa^2\beta + \cos^2\gamma = 1$

This is the universal relation connecting the direction cosines of every line.

Sometimes the direction cosines of a line are denoted by l, m, n (i.e., $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$) and then the above relation becomes $l^2 + m^2 + n^2 = 1$.

Note: If P has coordinate (x, y, z), then the line OP has direction cosines l, m, n and OP = k, then we have, x = lk, y = mk, z = nk.

In particular if OP = 1, P has coordinate (l, m, n).

6.8 DIRECTION RATIO OF A LINE

A set of three numbers which are proportional to the direction cosines of a line are called direction ratios(or direction numbers). If a, b, c are direction ratios of a line, so also are ka, kb, kc where k is any non zero constant.

If direction ratios of a line are given, we can easily find its actual direction cosines.

Let l, m, n be the direction cosines of a line whose direction ratios are given to be a, b, c.

Then, $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=k$ (say), Where, l = ak, m = bk, n = ck

But we have, $l^2 + m^2 + n^2 = 1$

$\Rightarrow (ak)^2 + (bk)^2 + (ck)^2 = 1$

$\Rightarrow k^2(a^2 + b^2 + c^2) = 1$

$\Rightarrow k^2 = \frac{1}{a^2+b^2+c^2}$

$\Rightarrow k = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$

Hence, $l = \frac{a}{\sqrt{a^2+b^2+c^2}}$, $m = \frac{b}{\sqrt{a^2+b^2+c^2}}$, $n = \frac{c}{\sqrt{a^2+b^2+c^2}}$

or,

$l = -\frac{a}{\sqrt{a^2+b^2+c^2}}$, $m = -\frac{b}{\sqrt{a^2+b^2+c^2}}$, $n = -\frac{c}{\sqrt{a^2+b^2+c^2}}$

Hence the direction cosines of a line having direction ratios a, b, c are,

$\left(\frac{a}{\rho},\frac{b}{\rho},\frac{c}{\rho}\right)$ or, $\left(-\frac{a}{\rho},-\frac{b}{\rho},-\frac{c}{\rho}\right)$, where, $\rho=\sqrt{a^2+b^2+c^2}$.

In fact two directed lines are associated with a line having direction ratios a, b, c one of which has direction cosines $\frac{a}{\rho},\frac{b}{\rho},\frac{c}{\rho}$ and the other has $-\frac{a}{\rho},-\frac{b}{\rho},-\frac{c}{\rho}$. Usually we shall be concerned with a line without any consideration of direction and take $\frac{a}{\rho},\frac{b}{\rho},\frac{c}{\rho}$ as its direction cosines.

For example if a line has direction ratios 2, –2, 1, then its direction cosines are,

$$\frac{2}{\sqrt{2^2+(-2)^2+1^2}}, -\frac{2}{\sqrt{2^2+(-2)^2+1^2}}, \frac{1}{\sqrt{2^2+(-2)^2+1^2}}$$

i.e., $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$.

Direction ratios of the line joining two given points:

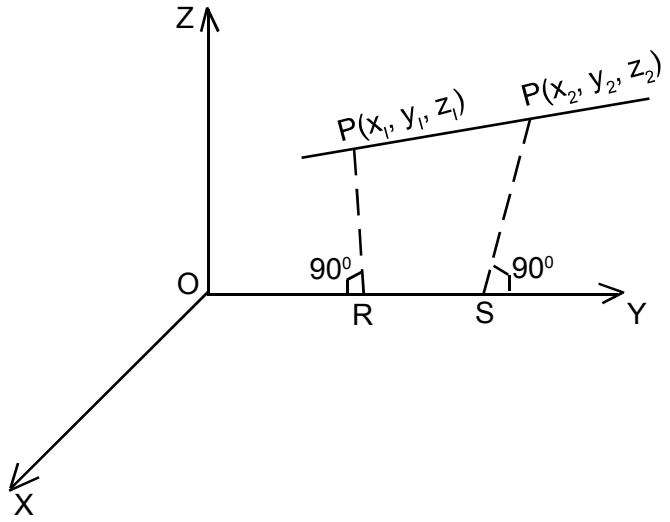


Fig. 6.9

Consider the line passing through the two given points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and suppose that α, β, γ are the angles made by this line with the positive directions of the axes of x, y, z respectively. Draw PR and QS perpendicular to y-axis; note that in general PR and QS are skew (i.e. non-coplanar) lines.

Now, RS = projection of the segment PQ on OY

= PQcosβ (β is the angle made by line PQ with OY)

= kcosβ (when k = PQ)

but, RS = OS – OR = $y_2 - y_1$

$\therefore$ kcosβ = $y_2 - y_1$

Similarly projecting PQ on OX and OZ respectively, we get.

kcosα = $x_2 - x_1$ and kcosγ = $z_2 - z_1$

Thus we get, $\frac{\cos\alpha}{x_2 - x_1} = \frac{\cos\beta}{y_2 - y_1} = \frac{\cos\gamma}{z_2 - z_1} = \frac{1}{k}$

It follows that the line passing through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ has direction ratios $x_2 - x_1, y_2 - y_1, z_2 - z_1$. Moreover the actual direction cosines of the line are, $\frac{x_2 - x_1}{k}, \frac{y_2 - y_1}{k}, \frac{z_2 - z_1}{k}$

Where, $k^2 = PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$.

For example, suppose that a line passes through the points (2, 0, –1) and (–3, 2, 7). Then its direction ratios are –3 – 2, 2 – 0, 7 – (–1), i.e., –5, 2, 8; also as, $k = \sqrt{(-5)^2 + 2^2 + 8^2} = \sqrt{93}$,

then direction cosines of the line are, $-\frac{5}{\sqrt{93}}, \frac{2}{\sqrt{93}}, \frac{8}{\sqrt{93}}$.

Angle between two lines whose direction cosines are given:

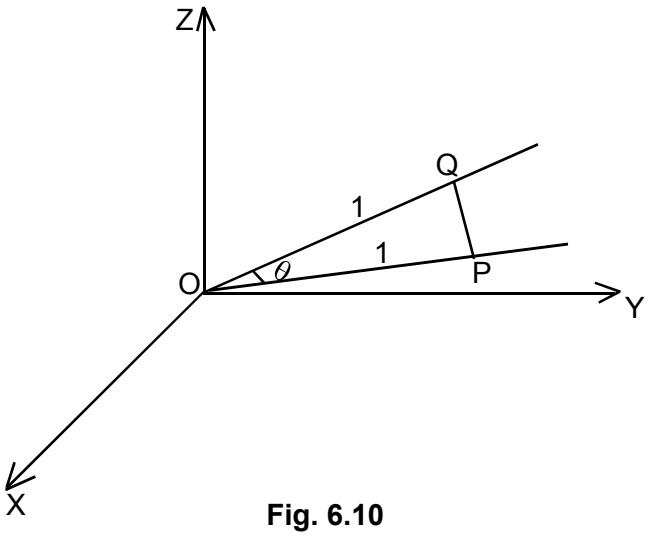


Fig. 6.10

Let θ be the required angle between two lines in space whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2.

Through the origin O draw lines parallel to the above two lines and take points P and Q on them such that OP = OQ = 1. Then the angle between OP and OQ is θ. As line OP has direction cosines l_1, m_1, n_1 and OP = 1, P has coordinates (l_1, m_1, n_1). Again line OQ has direction cosines l_2, m_2, n_2 and OQ = 1 so that Q has coordinates (l_2, m_2, n_2).

Hence, $PQ^2 = (l_1 - l_2)^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2$

$= (l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) - 2(l_1l_2 + m_1m_2 + n_1n_2)$

$= 1 + 1 - 2(l_1l_2 + m_1m_2 + n_1n_2)$

i.e., $PQ^2 = 2[1 - (l_1l_2 + m_1m_2 + n_1n_2)]$

Again from ΔOPQ, by elementary trigonometry,

$PQ^2 = OP^2 + OQ^2 - 2OP.OQ\cos\angle POQ$

i.e., $PQ^2 = 1^2 + 1^2 - 2.1.1\cos\theta$

i.e., $PQ^2 = 2(1 - \cos\theta)$

Hence we get, $2(1 - \cos\theta) = 2[1 - (l_1l_2 + m_1m_2 + n_1n_2)]$

$\therefore \cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

Corollary 1: $\sin\theta = \pm\sqrt{\sum(m_1n_2 - m_2n_1)^2}$ because,

$\sin^2\theta = 1 - \cos^2\theta = 1.1 - \cos^2\theta$

$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - 2(l_1l_2 + m_1m_2 + n_1n_2)^2$

$= (m_1^2n_2^2 + m_2^2n_1^2 - 2m_1m_2n_1n_2) + (n_1^2l_2^2 + n_2^2l_1^2 - 2n_1n_2l_1l_2)$

$+ (l_1^2m_2^2 + l_2^2m_1^2 - 2l_1l_2m_1m_2)$

$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$

$= \sum(m_1n_2 - m_2n_1)^2$

Corollary 2: Condition for parallelism of the two lines,

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \pm 1$$

either, $l_1 = l_2$, $m_1 = m_2$, n_2 or, $l_1 = -l_2$, $m_1 = -m_2$, $n_1 = -n_2$ because for parallelism we must have $\theta = 0$ or, $\theta = \pi$.

Corollary 3: Condition for perpendicularity of two lines is $l_1l_2 + m_1m_2 + n_1n_2 = 0$, because in such a case $\theta = \frac{\pi}{2}$, i.e., $\cos\theta$.

Angle between two lines whose direction ratios are given: Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of two lines in space. Then their direction cosines are $\frac{a_1}{k_1}, \frac{b_1}{k_1}, \frac{c_1}{k_1}$ and $\frac{a_2}{k_2}, \frac{b_2}{k_2}, \frac{c_2}{k_2}$,

respectively where, $k_1 = \sqrt{a_1^2 + b_1^2 + c_1^2}$ and $k_2\ \sqrt{a_2^2 + b_2^2 + c_2^2}$.

Hence the angle θ between the two given lines is given by

$$\cos\theta = \frac{a_1}{k_1}.\frac{a_2}{k_2} + \frac{b_1}{k_1}.\frac{b_2}{k_2} + \frac{c_1}{k_1}.\frac{c_2}{k_2}$$

$$= \frac{a_1a_2 + b_1b_2 + c_1c_2}{k_1k_2}$$

i.e., $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

i.e., $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\sum a_1^2}\sqrt{\sum a_2^2}}$

Corollary 1: $\sin\theta = \pm\dfrac{\sqrt{(b_1c_2 - b_2c_1)^2}}{\sqrt{\sum a_1^2}\sqrt{\sum a_2^2}}$

Corollary 2: Condition for parallelism is, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Corollary 3: Condition for perpendicularity is, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

ILLUSTRATIVE EXAMPLES:

Example 1: Can the numbers $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ be the direction cosines of a line?

Solution: If possible, let there be angles α, β, γ, such that,

$$\cos\alpha = \frac{1}{2},\ \cos\beta = \frac{1}{3},\ \cos\gamma = \frac{1}{4};$$

but then, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \neq 1$

Hence, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ cannot be the direction cosines of a line.

Example 2: Find the direction cosines of the line passing through the two points (1, –1, 2) and (2, 0, 1).

Solution: The line passing through the two points (1, –1, 2) and (2, 0, 1) has direction ratios 2 – 1, 0 – (–1), 1 – 2, i.e., 1, 1, –1.

As $\sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$,

we see that the direction cosines of the line are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$.

Example 3: Given the points A(1, –2, 1), B(0, 1, 4), C(–3, 1, 2) and D(1, 2, 3), find the angle between the lines AB and CD.

Solution: The line AB has direction ratios 0 – 1, 1 – (–2), 4 – 1,

i.e., –1, 3, 3.

The line CD has direction ratios 1 – (–3), 2 – 1, 3 – 2, i.e., 4, 1, 1.

If θ be the required angle between the lines AB and CD, we have,

$$\cos\theta = \frac{(-1).4 + 3.1 + 3.1}{\sqrt{(-1)^2 + 3^2 + 3^2}\sqrt{4^2 + 1^2 + 1^2}}$$

$$= \frac{-4 + 3 + 3}{\sqrt{19}\sqrt{18}}$$

$$= \frac{2}{\sqrt{19} \times 3\sqrt{2}}$$

$$= \frac{\sqrt{2}}{3\sqrt{19}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{2}}{3\sqrt{19}}\right)$$



CHECK YOUR PROGRESS

Q.5: If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of a line, prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

Q.6: A line makes angles of 30^0 and 60^0 with the axes of x and y respectively. What angle does it make with the z-axis?

Q.7: Find the angle between the lines having direction ratios –1, 2, 1 and 2, 3, –1.

Q.8: Show that the line joining the points (2, 1, –6) and (4, –3, 2) is parallel to the line joining the points (2, 2, –3) and (5, –4, 9).

Q.9: Given the points A(1, –1, 2), B(3, 2, –1) and C(0, 1, 2), find the angle between the lines OA and BC.

Q.10: A line has direction ratios 1, –2, 3 and the point P moves in such a way that OP is always perpendicular to the line. Find the equation of the locus of P.



6.10 LET US SUM UP

- The three coordinate planes divide the entire space into eight parts each of which is called an octant and the signs of the coordinates of a point determine the octant in which it lies.
- A line in space makes angles α, β, γ with the positive directions of x-axis, y-axis, z-axis repectively, then α, β, γ are called the direction angles of the line and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the line.
- A set of three numbers which are proportional to the direction cosines of a line are called direction ratios(or direction numbers).



6.11 FURTHER READING

1) Jain, P. K. and Ahmhad, K. *Textbook of Analytical Geometry of Three Dimensions*. New Delhi: New Age International (P) Ltd.
2) Khan, R. M. *Analytical Geometry Of Two & Three Dimensions & Vector Analysis*. New Central Book Agency (P) Limited.
3) Kishan, Hari. *Coordinate Geometry of Two dimensions*. New Delhi: Atlantic Publishers & Distributors.
4) Pal, Sat and Pal, Harbans. *A Textbook of Two Dimensional Geometry*. New Age International Publishers.
5) Sharma, G. C. and Jain, Madhu. *Coordinate Geometry (2D and 3D)*. New Delhi: Galgotia Publications Pvt Ltd.
6) Sharma, A. K. *Coordinate Geometry (2D)*. New Delhi: Discovery Publishing House.
7) Vasistha, A. R., Agrawal, D. C. *Analytical Geometry of Three Dimensions (Coordinate of Solid Geometry)*.

6.12 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: a) For the point (–2, –1, 5) $x = -2 < 0$, $y = -1 < 0$, $z = 5 > 0$ and so the point (–2, –1, 5) lies in the octant $OX'Y'Z$ (octant III).

b) For the point (1, 1, 7) all the coordinates are positive and so the point lies in the octant OXYZ (octant I).

c) For the point (–1, –1, –2), x, y and z are all negative so that the point lies in octant $OX'Y'Z'$ (octant VIII).

Ans. to Q. No. 2: Let $P(\alpha, \beta, \gamma)$ be the centre of the sphere which passes through the points O(0, 0, 0), A(a, 0, 0), B(0, b, 0) and C(0, 0, c) .

Then, PO = PA = PB = PC (each = radius of the sphere)

$\Rightarrow PO^2 = PA^2 = PB^2 + PC^2$

$\Rightarrow \alpha^2, \beta^2 + \gamma^2 = (\alpha - a)^2 + \beta^2 + \gamma^2 = \alpha^2 + (\beta - b)^2 + \gamma^2 =$

$\alpha^2 + \beta^2 + (\gamma - c)^2$

$\Rightarrow 0 = -2a\alpha + a^2 = -2b\beta + b^2 = -2c\gamma + c^2$

Now, $0 = -2a\alpha + a^2 \Rightarrow 2a\alpha = a^2 \Rightarrow \alpha = \frac{a}{2}$

$0 = -2a\beta + b^2 \Rightarrow 2b\beta = b^2 \Rightarrow \beta = \frac{b}{2}$

$0 = -2c\gamma + c^2 \Rightarrow 2c\gamma = c^2 \Rightarrow \gamma = \frac{c}{2}$

Hence the centre of the sphere is the point (α, β, γ), i.e., $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

Also the radius of the sphere,

$$PO = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2} = \frac{1}{2}\sqrt{a^2 + b^2 + c^2}\,.$$

Ans. to Q. No. 3: Let D be the middle point of the side BC; then D has coordinates $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2}\right)$.

The centroid G of the triangle ABC divides the median AD internally in the ratio 2 : 1; so the coordinates of G are,

$$\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+1x_1}{2+1}, \frac{2\left(\frac{y_2+y_3}{2}\right)+1y_1}{2+1}, \frac{2\left(\frac{z_2+z_3}{2}\right)+1z_1}{2+1} \right)$$

i.e., $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3} \right)$.

Ans. to Q. No. 4: Suppose that the line joining the points P(–2, 4, 7) and Q(3, –5, 8) meets the x-coordinate plane (i.e., the yz-plane) at R(x, y, z);

Also let, PR : RQ = λ : 1.

Then, $x = \frac{\lambda.3+(-2)}{\lambda+1} = \frac{3\lambda-2}{\lambda+1}$,

$y = \frac{\lambda.(-5)+4}{\lambda+1} = \frac{4-5\lambda}{\lambda+1}$,

$z = \frac{\lambda.8+7}{\lambda+1} = \frac{8\lambda+7}{\lambda+1}$

As R(x, y, z) lies in the x-coordinate plane, we must have,

$x = 0$

$\Rightarrow \frac{3\lambda-2}{\lambda+1} = 0$

$\Rightarrow 3\lambda - 2 = 0 \Rightarrow 3\lambda - 2 = 0$

$\Rightarrow \lambda = \frac{2}{3}$

Hence, $y = \frac{4-5\lambda}{\lambda+1} = \frac{4-5\times\frac{2}{3}}{\frac{2}{3}+1} = \frac{2}{5}$

and $z = \frac{8\lambda+7}{\lambda+1} = \frac{8\times\frac{2}{3}+7}{\frac{2}{3}+1} = \frac{37}{5}$.

$\therefore$ The line joining the given points P and Q meets the x-coordinate plane at the point $\left(0, \frac{2}{5}, \frac{37}{5}\right)$.

Ans. to Q. No. 5: We have, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 0$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$

Ans. to Q. No. 6: A line makes angle $\alpha = 30^0$ with x-axis and $\beta = 60^0$ with the y-axis; to find the angle γ made by the line with the z-axis.

We have, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \cos^2 30^0 + \cos^2 60^0 + \cos^2\gamma = 1$

$\Rightarrow \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$

$\Rightarrow 1 + \cos^2\gamma = 1$

$\Rightarrow \cos^2\gamma = 0$

$\Rightarrow \cos\gamma = 0$

$\Rightarrow \gamma = 90^0$

$\therefore$ The line makes an angle of 90^0 with the z-axis.

Ans. to Q. No. 7: Let θ be the angle between the lines having direction ratios –1, 2, 1 and 2, 3, –1.

Then, $\cos\theta = \frac{(-1)(2)+(2)(3)+(1)(-)}{\sqrt{(-1)^2+2^2+1^2}\sqrt{2^2+3^2+(-1)^2}}$

$= \frac{-2+6-1}{\sqrt{6}\sqrt{14}}$

$= \frac{3}{\sqrt{84}}$

$= \frac{3}{2\sqrt{21}}$

$\therefore \theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$, which is the required angle.

Ans. to Q. No. 8: The line joining the points (2, 1, –6) and (4, –3, 2) has direction ratios 4 – 2, (–3) – 1, 2 – (–6), i.e., 2, –4, 8.

The line joining the points (2, 2, –3) and (5, –4, 9) has direction ratios, 5 – 2, (–4) – 2, 9 – (–3), i.e., 3, –6, 12.

We note that, $\frac{2}{3} = \frac{-4}{-6} = \frac{8}{12}$ and hence the two lines are parallel.

Ans. to Q. No. 9 The line OA passing through O(0, 0, 0) and A(1, –1, 2) has direction ratios 1 – 0, (–1) – 0, 2 – 0, i.e., 1, –1, 2;

Again the line BC passing through B(3, 2, –1) and C(0, 1, 2) has direction ratios 0 – 3, 1 – 2, 2 – (–1), i.e., –3, –1, 3.

Hence the angle θ between the lines OA and BC is given by

$$\cos\theta = \frac{1.(-3)+(-1).(-1)+2.3}{\sqrt{1^2+(-1)^2+2^2}\sqrt{(-3)^2+(-1)^2+3^2}}$$

$$= \frac{4}{\sqrt{6}\sqrt{19}}$$

So that, $\theta = \cos^{-1}\left(\frac{4}{\sqrt{114}}\right)$

Ans. to Q. No. 10: Let (x, y, z) be the coordinates of the moving point P.As the line OP having direction ratios x, y, z is always perpendicular to the line having direction ratios 1, –2, 3.

We must have, x.1 + y.(–2) + z.3 = 0

i.e., x – 2y + 3z = 0

This is the required equation to the locus of P.



6.13 MODEL QUESTIONS

Q.1: Find the ratio in which the line joining the points (2, 4, 5), (3, 5, –4) is divided by the XY-plane.

Q.2: A line makes angles of 45^0 and 60^0 with the positive axes of x and y respectively. What angle does it make with the positive axis of z?

Q.3: What are the direction cosines of the lines equally inclined to the axes? How many such lines are there?

Q.4: Find the direction cosines of the line joining the points P(4, 3, –5) and Q(–2, 1, –8).

Q.5: P, Q, R, S are the points (–2, 3, 4), (–4, 4, 6), (4, 3, 5) and (0, 1, 2). Find the direction cosines of PQ and RS.

Q.6: The direction ratios of a line are 6, 2, 3. Find the direction cosines.

Q.7: Find the angle between the lines whose direction cosines are proportional to 1, 2, 1 and 2, –3, 6.

Q.8: Find the direction cosines of a line perpendicular to the two lines whose direction ratios are 1, 2, 3 and –2, 1, 4.

Q.9: Show that the join of points (1, 2, 3) and (4, 5, 6) is parallel to the join of the points (–4, 3, –6) and (2, 9, 2).

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UNIT 7: THE PLANE

UNIT STUCTURE

7.1 Learning Objectives

7.2 Introduction

7.3 The Plane

7.4 Let Us Sum Up

7.5 Further Reading

7.6 Answers to Check Your Progress

7.7 Model Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know about the plane.

7.2 INTRODUCTION

In previous unit, we have discussed about the co-ordinate system and co-ordinates of a point in space, distance between two points in space and position of a point dividing the join of two points in a given ratio internally/ externally. In this,we will discuss equation of plane and various equations of plane.

7.3 THE PLANE

The General Equation of a Plane: The general equation of the first degree in x, y, z, viz. Ax + By + Cz + D = 0, where, A, B, C and D are constants and A, B, C are not all zero,always represents a plane.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on the locus represented by the equation,

$Ax + By + Cz + D = 0$ ... ...(i)

Then we have,

$Ax_1 + By_1 + Cz_1 + D = 0$ ... ...(ii)

and $Ax_2 + By_2 + Cz_2 + D = 0$ ... ...(iii)

Multiplying (iii) by k, where, k is an arbitrary constant and then adding it to (ii), we get,

$$A(kx_2 + x_1) + B(ky_2 + y_1) + C(Kz_2 + z_1) + D(k + 1) = 0$$

$$\Rightarrow A\left(\frac{kx_2 + x_1}{k+1}\right) + B\left(\frac{ky_2 + y_1}{k+1}\right) + C\left(\frac{kz_2 + z_1}{k+1}\right) + D = 0 \quad (k \neq -1)$$

Whence it follows that the point $\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$ lies on the locus represented by (i), whatever by the value of $k(\neq -1)$. But this is a point lying on the line joining P and Q and dividing the segment PQ in the ratio k : 1 and by giving all possible real value to k (different from –1), we see that every point on the line PQ lies on the locus represented by (i). So the line PQ, where, P and Q are any two points on the locus of (i), lies wholly on the locus of (i). This shows that (i) represents a plane.

Note:

i) We have already seen in 6.3 that the equations of the three coordinate planes are x = 0, y = 0 and z = 0 which are special cases of the general equation with A = 1 and B = C = D = 0, B = 1 and A = C = D = 0, and C = 1 and A = B = D = 0 respectively.

ii) The general equation of a plane, viz, Ax + By + Cz + D = 0 apparently contains four arbitrary constants; however as at least one of the three constants A, B, C is non zero, we can divide the equation throughout by a non zero constant and the resulting equation then contains three independent arbitrary constants. For example if $A \neq 0$, dividing by A, the equation becomes,

$$x + \left(\frac{B}{A}\right)y + \left(\frac{C}{A}\right)z + \frac{D}{A} = 0$$

i.e. $x + B'y + C'z + D' = 0 \quad \left(B' = \frac{B}{A}, C' = \frac{C}{A}, D' = \frac{D}{A}\right)$

Which contains three arbitrary constants B', C' and D'.

It is clear that three independent conditions are necessary to determine a particular plane.

iii) If the plane Ax + By + Cz + D = 0 passes through the point (x_1, y_1, z_1), then we must have,

$Ax_1 + By_1 + Cz_1 + D = 0$

$\Rightarrow D = -(Ax_1 + By_1 + Cz_1)$

Then the equation of the plane becomes,

$Ax + By + Cz - (Ax_1 + By_1 + Cz_1) = 0$

i.e., $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

This is the equation of any plane passiong through the point (x_1, y_1, z_1). In particular $Ax + By + Cz = 0$ represents any plane passing through the origin. It is easy to see that $Ax + By = 0$ represents a plane containing the z-axis; similarly for $By + Cz = 0$ and $Ax + Cz = 0$.

Equation to a plane in the intercept form:

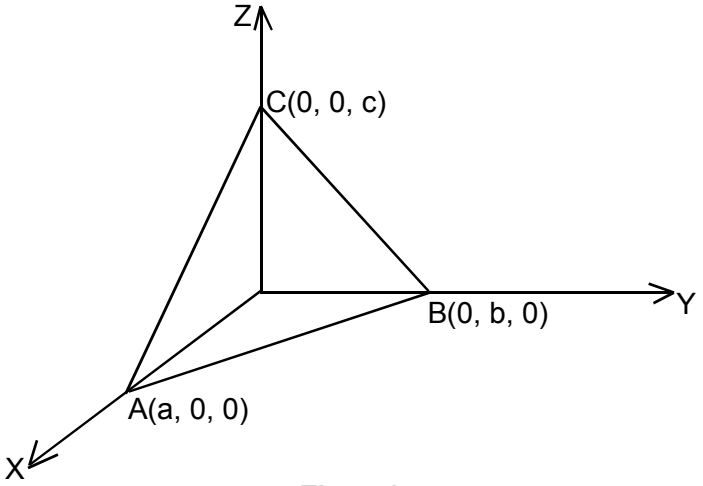


Fig. 7.1

Suppose that a plane makes intercepts OA = a on x-axis, OB = b on y-axis and OC = c on z-axis. This implies that the plane passes through the points A(a, 0, 0), B(0, b, 0) and C(0, 0, c).

Let the required equation of the plane (which will be a first degree equation in x, y, z) be

$lx + my + nz = p$ ... ...(1)

As it passes through the points A(a, 0, 0), B(0, b, 0) and C(0, 0, c), we must have, $la = p$, $mb = p$, $nc = p$

Hence, (1) becomes $\frac{p}{a}.x + \frac{p}{b}y + \frac{p}{c}z = p$

i.e., $\frac{x}{a}. + \frac{y}{b} + \frac{z}{c} = 1$

Which is the required equation of the plane.

Note:

i) If a plane cuts the x-axis at A(a, 0, 0) and is parallel to the yz-plane, then both b and c are infinite and the above equation reduces to $\frac{x}{a} = 1$, i.e., x = a. Thus the equation x = a represents a plane parallel to the yz-plane (or x-plane) and it meets the x-axis at (a, 0, 0). Likewise y = b represents a plane parallel to the zx-plane (or y-plane) meeting the y-axis at (0, b, 0) and z = c represents a plane parallel to the xy-plane (or z-plane) meeting the z-axis at (0, 0, c).

ii) The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ will be parallel to the x-axis if a is infinite and in that event the equation becomes $\frac{y}{b} + \frac{z}{c} = 1$, which is of the form By + Cz = 1; so, the equation of a plane parallel to the x-axis is of the form By + Cz = 1. Likewise the equation of the plane parallel to the y-axis is of the form Ax + Cz = 1 and that of a plane parallel to the z-axis is of the form Ax + By = 1.

iii) An equation of the form By + Cz + D = 0 represents a plane parallel to the x-axis if $D \neq 0$ and a plane through the x-axis if, D = 0. Similarly for Ax + Cz + D = 0 and Ax + By + D = 0.

Equation to a plane in the normal form:

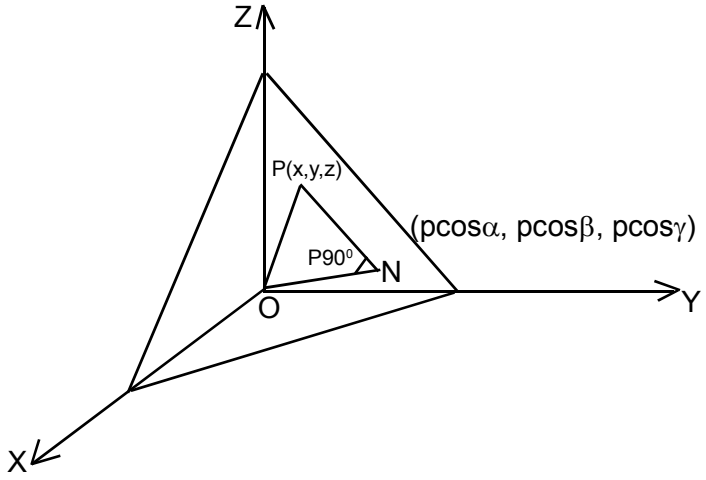


Fig. 7.2

Let ON = p (p > 0) be the normal from the origin to a given plane and let its direction cosines be $\cos\alpha$, $\cos\beta$, $\cos\gamma$. Then N has coordinates ($p\cos\alpha$,

$p\cos\beta$, $p\cos\gamma$). Clearly the line PN has direction ratios $x - p\cos\alpha$, $y - p\cos\beta$, $z - p\cos\gamma$. As ON $\perp$ PN, we have,

$$\cos\alpha(x - p\cos\alpha) + \cos\beta(y - p\cos\beta) + \cos\gamma(z - p\cos\gamma) = 0$$

i.e., $x\cos\alpha + y\cos\beta + z\cos\gamma = p(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$

i.e., $x\cos\alpha + y\cos\beta + z\cos\gamma = p$

This is the required equation of the given plane,known as the normal form of the equation of the plane.

Equation of a plane passing through three given (non colinear) points: Suppose that a plane passes through three given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3); to find the equation of the plane.

As the required plane passes through the point (x_1, y_1, z_1), its equation will be of the form,

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \qquad \text{... ... ...(i)}$$

As the plane passes also through the points (x_2, y_2, z_2) and (x_3, y_3, z_3), we have,

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \qquad \text{... ... ...(ii)}$$

and $a(x_3 - x_1) + b(y_3 - y_1) + c(z_3 - z_1) = 0 \qquad \text{... ... ...(iii)}$

Eliminating a, b, c from (i), (ii) and (iii), we get,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

as the required equation of the plane.

Reduction of the general equation of a plane to the intercept form and the normal form,

i) Consider the general equation of a plane $Ax + By + Cz + D = 0$.

This is equivalent to $Ax + By + Cz = -D$

$$\text{i.e., } -\frac{A}{D}x - \frac{B}{D}y - \frac{C}{D}z = 1 \quad (D \neq 0)$$

$$\text{i.e., } \frac{x}{-\frac{D}{A}} + \frac{y}{-\frac{D}{B}} + \frac{z}{-\frac{D}{C}} = 1$$

Which is the intercept form of the equation of the plane. The plane makes intercepts $-\frac{D}{A}, -\frac{D}{B}, -\frac{D}{C}$ on the axes of x, y, z respectively.

ii) Let the normal form of the equation of the plane,

$Ax + By + Cz + D = 0$

be, $x\cos\alpha + y\cos\beta + z\cos\gamma = p \ (p > 0)$

i.e., $x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0$

Then we must have,

$$\frac{\cos\alpha}{A} = \frac{\cos\beta}{B} = \frac{\cos\gamma}{C} = \frac{-p}{D}$$

$$\Rightarrow \cos\alpha = -\frac{Ap}{D}, \cos\beta = -\frac{Bp}{D}, \cos\gamma = \frac{-Cp}{D}$$

But, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

So we get, $\left(-\frac{Ap}{D}\right)^2 + \left(-\frac{Bp}{D}\right)^2 + \left(\frac{-Cp}{D}\right)^2 = 1$

$$\Rightarrow p^2 (A^2 + B^2 + C^2) = D^2$$

$$\Rightarrow p^2 = \frac{D^2}{A^2 + B^2 + C^2}$$

$$\Rightarrow p = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}} \ (p > 0)$$

If $D > 0$, we get, $p = \frac{D}{\sqrt{A^2 + B^2 + C^2}}$, and

$$\cos\alpha = -\frac{Ap}{D} = -\frac{A}{\sqrt{A^2 + B^2 + C^2}}$$

$$\cos\beta = -\frac{Bp}{D} = -\frac{B}{\sqrt{A^2 + B^2 + C^2}}$$

$$\cos\gamma = -\frac{Cp}{D} = -\frac{C}{\sqrt{A^2 + B^2 + C^2}}$$

Hence the equation of the plane $Ax + By + Cz + D = 0$, $D > 0$, in the normal form is, $x\cos\alpha + y\cos\beta + z\cos\gamma = p$,

i.e., $$\left(-\frac{A}{\sqrt{A^2 + B^2 + C^2}}\right)x + \left(-\frac{B}{\sqrt{A^2 + B^2 + C^2}}\right)y + \left(-\frac{C}{\sqrt{A^2 + B^2 + C^2}}\right)z$$

$$= \frac{D}{\sqrt{A^2 + B^2 + C^2}}$$

If D < 0, then, $p = \frac{|D|}{\sqrt{A^2+B^2+C^2}} = \frac{-D}{\sqrt{A^2+B^2+C^2}}$ and the normal form of equation of the plane is,

$$\left(\frac{A}{\sqrt{A^2+B^2+C^2}}\right)x+\left(\frac{B}{\sqrt{A^2+B^2+C^2}}\right)y+\left(\frac{C}{\sqrt{A^2+B^2+C^2}}\right)z$$

$$= -\frac{D}{\sqrt{A^2+B^2+C^2}}$$

Remarks:

i) From the above discussion we get the following working rule for reducing the equation of a plane to its normal form:

Let the equation of the plane be Ax + By + Cz + D = 0, Take the constant term at the right hand side with postive sign;

–Ax – By – Cz = D,

then divide the equation throughout by $\sqrt{A^2+B^2+C^2}$ giving us,

$$\left(-\frac{A}{\sqrt{A^2+B^2+C^2}}\right)x+\left(-\frac{B}{\sqrt{A^2+B^2+C^2}}\right)y+\left(-\frac{C}{\sqrt{A^2+B^2+C^2}}\right)z$$

$$= \frac{D}{\sqrt{A^2+B^2+C^2}}$$

Which is the required normal form of the equation of the plane.

For example let us reduce the plane 2x – 3y + 4z – 11 = 0 to its normal form: 2x – 3y + 4z – 11 = 0

i.e., 2x – 3y + 4z = 11

(the constant term on the right side is postive)

Dividing both sides by $\sqrt{2^2+(-3)^2+4^2}$, i.e., $\sqrt{29}$, we get,

$$\frac{2}{\sqrt{29}}x-\frac{3}{\sqrt{29}}y+\frac{4}{\sqrt{29}}z=\frac{11}{\sqrt{29}}$$

Which is the required normal form of the given plane.

ii) We have seen that, $\frac{\cos\alpha}{A}=\frac{\cos\beta}{B}=\frac{\cos\gamma}{C}$ but, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal to the plane Ax + By + Cz + D = 0. It follows that the numbers A, B, C (coefficients of x, y, z) are direction ratios of the normal to the plane Ax + By + Cz + D = 0.

Also, $p = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}$ is the perpendicular distance of the plane from the origin.

Angle between two planes whose equations are given:

Let θ be the angle between the two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$.

Then θ is also the angle between the normals to the two planes having A_1, B_1, C_1 and A_2, B_2, C_2 as direction ratios respectively, we have,

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

giving the required angle θ between the two planes.

Remarks:

1) The two planes will be at right angles if $\theta = \frac{\pi}{2}$, i.e., if $\cos\theta = 0$,

   i.e., $A_1A_2 + B_1B_2 + C_1C_2 = 0$.

   This is the condition of perpendicularity of the two planes,

   $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$.

2) If the two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ parallel, a normal to one of them is also a normal to the other. As normal to the first plane has direction ratios A_1, B_1, C_1 and that to the second has direction ratios A_2, B_2, C_2, we must have, $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$, which is the condition of parallelism of the two planes.

3) Clearly the two planes $Ax + By + Cz + D = 0$ and $Ax + By + Cz + D' = 0$ are parallel. Thus the equation of any plane parallel to a given plane $Ax + By + Cz + D = 0$ is $Ax + By + Cz + D' = 0$; they differ only in the constant term.

Perpendicular distance of a point from a plane:

Let, $P(x', y', z')$ be a given point and $N(\alpha, \beta, \gamma)$ be the foot of the perpendicular drawn from P to the plane $Ax + By + Cz + D = 0$.

The line PN has direction ratios $\alpha - x'$, $\beta - y'$, $\gamma - z'$. But PN is a normal to the plane $Ax + By + Cz + D = 0$, so that A, B, C are its direction ratios.

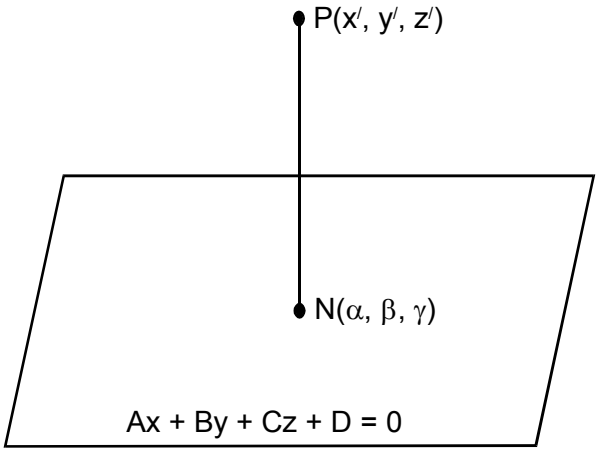


Fig. 7.3

Then, $\alpha - x'$, $\beta - y'$, $\gamma - z'$ and A,B,C are two sets of direction ratios of the same line PN and so we have,

$$\frac{\alpha - x'}{A} = \frac{\beta - y'}{B} = \frac{\gamma - z'}{C} = k \quad \text{(say);}$$

then, $\alpha = Ak + x'$, $\beta = Bk + y'$, $\gamma = Ck + z'$.

As $N(\alpha, \beta, \gamma)$ lies on the plane $Ax + By + Cz + D = 0$, we get,

$A\alpha + B\beta + C\gamma + D = 0$

$\Rightarrow A(Ak + x') + B(Bk + y') + C(Ck + z') + D = 0$

$\Rightarrow (A^2 + B^2 + C^2)k = -(Ax' + By' + Cz' + D)$

$$\Rightarrow k = -\frac{Ax' + By' + Cz' + D}{A^2 + B^2 + C^2}$$

$\therefore PN^2 = (\alpha - x')^2 + (\beta - y')^2 + (\gamma - z')^2$

$= (Ak)^2 + (Bk)^2 + (Ck)^2$

$= (A^2 + B^2 + C^2)k^2$

$$= (A^2 + B^2 + C^2)\left(-\frac{Ax' + By' + Cz' + D}{A^2 + B^2 + C^2}\right)^2$$

i.e., $$PN^2 = \frac{(Ax' + By' + Cz' + D)^2}{A^2 + B^2 + C^2}$$

$\therefore$ Length of the perpendicular from $P(x', y', z')$ to the given plane = PN

$$= \left|\frac{Ax' + By' + Cz' + D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Corollary: Perpendicular distance from the origin

$$= \left| \frac{D}{\sqrt{A^2 + B^2 + C^2}} \right|$$

Equation of any plane passing through two given planes:

Given two planes $A_1x_1 + B_1y_1 + C_1z_1 + D_1 = 0$ ... ...(i)

and $A_2x_2 + B_2y_2 + C_2z_2 + D_2 = 0$ ... ...(ii)

the equation,

$(A_1x_1 + B_1y_1 + C_1z_1 + D_1) + k(A_2x + B_2y + C_2z + D_2) = 0$ ... ...(iii)

where, k is an arbitrary constant ,represents a plane passing through the line of intersection of the planes (i) and (ii) as explained below:

(iii) is an equation of first degree in x, y, z and so it represents a plane. Moreover the coordinates of any point on the line of intersection of the planes (i) and (ii) satisfy both equations (i) and (ii) and hence satisfy the equation (iii). Hence the planes represented by (iii) passes through the line of intersection of the planes (i) and (ii). As k is arbitrary, all such planes are given by (iii) by giving different values to k.Hence (iii) is the general equation of any plane passing through the line of intersection of the planes (i) and (ii).

ILLUSTRATIVE EXAMPLES:

Example 1: Find the direction cosines of the normal to the plane 2x + y – 2z = 12. What is the perpendicular distance of the plane from the origin?

Solution: The equation of the plane is 2x + y – 2z = 12 in which the coefficients of x, y and z are 2, 1 and –2 repectively. The normal to the plane has direction ratios 2, 1, –2. Noting that,

$$\sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3,$$

We see that the direction cosines of the normal to the plane are,

$$\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}.$$

Also perpendicular distance of the plane from the origin = $\frac{12}{3}$ = 4.

Example 2: Find the intercepts made by the plane 3x – 4y + 2z + 2 = 0 on the coordinate axes.

Solution: The equation of the plane is,

$3x - 4y + 2z + 2 = 0$

i.e., $3x - 4y + 2z = -2$

i.e., $\frac{3x}{-2} - \frac{4y}{-2} + \frac{2z}{-2} = \frac{-2}{-2}$

i.e., $\frac{x}{-\frac{2}{3}} + \frac{y}{\frac{1}{2}} + \frac{z}{-1} = 1$

$\therefore$ the intercepts made by the plane on the axes are $-\frac{2}{3}, \frac{1}{2}, -1$

Example 3: Find the equation of the plane passing through the points (0, 1, 1), (1, 1, 2) and (–1, 2, –2).

Solution: As the required plane passes through the point (0, 1, 1), its equation is of the form,

$A(x - 0) + B(y - 1) + C(z - 1) = 0$ ... ...(i)

As the plane also passes through the points (1, 1, 2) and (–1, 2, –2), their coordinates must satisfy the equation (i); so we get,

$A.1 + B.0 + C.1 = 0$ ... ...(ii)

and $A.(-1) + B.1 + C.(-3) = 0$ ... ...(iii)

Eliminating A, B and C from (i), (ii) and (iii), we get,

$$\begin{vmatrix} x & y-1 & z-1 \\ 1 & 0 & 1 \\ -1 & 1 & -3 \end{vmatrix} = 0$$

i.e., $x(0 - 1) - (y - 1)(-3 + 1) + (z - 1)(1 - 0) = 0$

i.e., $-x + 2(y - 1) + (z - 1) = 0$

i.e., $-x + 2y + z - 3 = 0$

This is the required equation of the plane.

Alternative Method:

The equation of the plane passing the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Hence the equation of the plane passing through the points (0, 1, 1), (1, 1, 2) and (–1, 2, –2) is,

$$\begin{vmatrix} x-0 & y-1 & z-1 \\ 1-0 & 1-1 & 2-1 \\ -1-0 & 2-1 & -2-1 \end{vmatrix} = 0$$

i.e., $\begin{vmatrix} x & y-1 & z-1 \\ 1 & 0 & 1 \\ -1 & 1 & -3 \end{vmatrix} = 0$

i.e., $x(0 - 1) - (y - 1)(–z + 1) + (z – 1)(1 – 0) = 0$

i.e., $–x + 2(y – 1) + (z – 1) = 0$

i.e., $–x + 2y + z – 3 = 0$

This is the required equation of the plane.

Example 4: Find the equation of the plane passing through the middle point of the join of the points (1, –2, 0) and (–1, 4, 2) and perpendicular to it.

Solution: The middle point of the join of the points (1, –2, 0) and (–1, 4, 2) is,

$$\left(\frac{1-1}{2}, \frac{-2+4}{2}, \frac{0+2}{2}\right)$$

i.e., (0, 1, 1).

Also the line joining them, having direction ratios –1 – 1, 4 – (–2), 2 – 0, i.e., –2, 6, 2 is perpendicular to the plane. So we have to find the equation of the plane which passes through the point (0, 1, 1) and whose normal has direction ratios –2, 6, 2.

Hence the required equation of the plane is,

$–2(x – 0) + 6(y – 1) + 2(z – 1) = 0$

i.e., $–2x + 6y – 6 + 2z – 2 = 0$

i.e., $–2x + 6y + 2z – 8 = 0$

i.e., $–2(x – 3y – z + 4) = 0$

i.e., $x – 3y – z = 0$

Example 5: Find the equation of the plane which is parallel to the y-axis and which passes through the points (1, –4, 2) and (2, 3, 1).

Solution: As the required plane is parallel to the y-axis,its equation is of the form $ax + cz + d = 0$ ... ...(i)

As it is to pass through the points (1, –4, 2) and (2, 3, 1), we get,

$a.1 + c.2 + d = 0$ ... ...(ii)

and $a.2 + c.2 + d = 0$ ... ...(iii)

Eliminating a, c and d from (i), (ii) and (iii), we get,

$$\begin{vmatrix} x & z & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

i.e., $x(2 – 1) – z(1 – 2) + 1(1 – 4) = 0$

i.e., $x + z – 3 = 0$

as the equation of the required plane.

Example 6: Find the equation of the plane through the point (2, 5, –8) perpendicular to both the planes $2x – 3y + 4z + 1 = 0$ and $4x + y – 2z + 6 = 0$.

Solution: Any plane through the point (2, 5, –8) is,

$a(x – 2) + b(y – 5) + c(z + 8) = 0$ ... ...(i)

If it is perpendicular to both the planes $2x – 3y + 4z + 1 = 0$ and $4x + y – 2z + 6 = 0$, then,

$a.2 + b.(–3) + c.4 = 0$ ... ...(ii)

and $a.4 + b.1 + c.(–2) = 0$ ... ...(iii)

[Note that the condition of perpendicularity $A_1A_2 + B_1B_2 + C_1C_2 = 0$ has been applied there.]

Eliminating a, b and c from (i), (ii) and (iii), we get,

$$\begin{vmatrix} x-2 & y-5 & z+8 \\ 2 & -3 & 4 \\ 4 & 1 & -2 \end{vmatrix} = 0$$

i.e., $(x – 2)(6 – 4) – (y – 5)(–4 – 16) + (z + 8)(2 + 12) = 0$

i.e., $2(x – 2) + 20(y – 5) + 14(z + 8) = 0$

i.e., $(x – 2) + 10(y – 5) + 7(z + 8) = 0$

i.e., $x + 10y + 7z + 4 = 0$ as the equation of the plane.

Example 7: Show that the two planes $4x – 6y + 2z – 1 = 0$ and $6x – 9y + 3z + 5 = 0$ are parallel. Also find the distance between them.

Solution: We know that the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel if, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

For the given two planes,

$4x - 6y + 2z - 1 = 0$ ... ...(i)

$6x - 9y + 3z + 5 = 0$ ... ...(ii)

We have, $\frac{a_1}{a_2} = \frac{4}{6} = \frac{2}{3}$, $\frac{b_1}{b_2} = \frac{-6}{-9} = \frac{2}{3}$, $\frac{c_1}{c_2} = \frac{2}{3}$

So that the condition $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ is satisfied.Hence the given planes (i) and (ii) are parallel.

Putting y = 0 and z = 0 in (i), we get, $4x = 1 \Rightarrow x = \frac{1}{4}$;

So, $\left(\frac{1}{4}, 0, 0\right)$ is a point on plane (i).

The distance between the parallel planes (i) and (ii) = the length of the perpendicul;ar drawn from the point $\left(\frac{1}{4}, 0, 0\right)$ on plane (i) to the plane (ii),

$$= \left|\frac{6.\frac{1}{4} - 9.0 + 3.0 + 5}{\sqrt{6^2 + (-9)^2 + 3^2}}\right|$$

$$= \left|\frac{\frac{3}{2} + 5}{\sqrt{126}}\right|$$

$$= \frac{\frac{13}{5}}{\sqrt{126}}$$

$$= \frac{13}{15\sqrt{14}}$$

$$= \frac{13\sqrt{14}}{210}$$

Example 8: Find the equation of the plane which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$ and is perpendicular to the plane $5x + 3y + 6z + 8 = 0$.

Solution: Any plane through the line of intersection of the given planes, $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$ is given by

$$x + 2y + 3z - 4 + \lambda(2x + y - z + 5) = 0$$

i.e., $(1 + 2\lambda)x + (2 + \lambda)y + (3 - \lambda)z + (5\lambda - 4) = 0$ ... ...(1)

This will be perpendicular to the plane $5x + 3y + 6z + 8 = 0$ if,

$$(1 + 2\lambda)5 + (2 + \lambda)3 + (3 - \lambda)6 = 0$$

$\Rightarrow 5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0$

$\Rightarrow 7\lambda = -29$

$\Rightarrow \lambda = -\frac{29}{7}$

putting this value of λ in (1), we get,

$$\left(1-\frac{58}{7}\right)x+\left(2-\frac{29}{7}\right)y+\left(3+\frac{29}{7}\right)z+\left(-\frac{145}{7}-4\right)=0$$

i.e., $-51x - 15y + 50z - 173 = 0$

i.e., $51x + 15y - 50z + 173 = 0$

as the required equation of the plane.

Example 9: Find the equation of the plane making intercepts $-\frac{1}{2}, 1, \frac{1}{2}$ on the coordinate axes. Reduce it to the normal form. Hence state the direction cosines of the normal to the plane and its perpendicular distance from the origin.

Solution: The equation of the plane is,

$$\frac{x}{-\frac{1}{2}}+\frac{y}{1}+\frac{z}{\frac{1}{2}}=1$$

i.e., $-2x + y + 2z = 1$

i.e., $2x - y - 2z + 1 = 0$ ... ...(1)

We now reduce (1) to the normal form, from (1)

$$2x - y - 2z = -1$$

i.e., $-2x + y + 2z = 1$

Dividing both sides by $\sqrt{(-2)^2+1^2+2^2}$, i.e., by 3, we get,

$$\left(-\frac{2}{3}\right)x+\frac{1}{3}y+\frac{2}{3}z=\frac{1}{3} \qquad \text{... ... ...(2)}$$

Which is the equation of the plane (1) in the normal form

$x\cos\alpha + y\cos x\beta + z\cos\gamma = p$

The direction cosines of the normal to the plane are,

$\cos\alpha, \cos\beta, \cos\gamma$, i.e., $-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$

and its perpendicular distance from the origin is p, i.e., $\frac{1}{3}$.



CHECK YOUR PROGRESS

Q.1: A plane cuts the axes in A, B, C and the centroid of the triangle ABC is (p, q, r). Find the equation of the plane.

Q.2: Find the equation of the plane passing through the points (1, 0, –1), (2, –3, 4) and (3, 1, 1). What intercepts does it make on the coordinate axes?

Q.3: Find the plane passing through the point (1, –4, 2) and the x-axis.

Q.4: Show that the planes x + 2y + 2z – 5 = 0 and 2x + y –2z = 0 are orthogonal. Find a plane through the point (2, –4, 7) perpendicular to both the above planes. What are the direction cosines of the normal to the last plane?

Q.5: Obtain the normal form of equation of the plane 3x + 2y – 6z + 14 = 0.

Q.6: Find the equation of the plane passing through the line of intersection of the planes x – y + 2z = 3 and 2x + 3y – z + 2 = 0 and the point (2, 0, –5).

Q.7: Find the angle between the planes x – 2y + 2z – 5 = 0 and 3x –2y + 6z = 0.

Q.8: Find the value of k so that the planes kx – y + 4z = 2 and 3x + 2ky – 11z = 8 may be orthogonal.



7.4 LET US SUM UP

- A first degree equation in x, y and z, viz. Ax + By + Cz + D = 0 represents a plane.The numbers A,B,C are direction ratios of normal to the plane.
- The equation of any plane passing the through the point (x_1, y_1, z_1) is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$, where A, B, C are direction ratios of its normal.
- The equation of a plane in the intercept form is $\frac{x}{a} + \frac{y}{b} + \frac{z}{a} = 1$, where, a, b, c are the intercepts made by the plane on the coordinate axes.
- The equation of a plane in the normal form is $x\cos\alpha + y\cos\beta + z\cos\gamma = p$ $(p > 0)$, where, p is the perpendicular distance of the plane from the origin and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the perpendicular.
- The equation of the plane passing through the three (non-collinear) points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- The angle between the two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is, $\cos^{-1} \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$
- The condition of parallelism of the two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is, $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$.
- The condition of perpendicularity of the two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is, $A_1A_2 + B_1B_2 + C_1C_2 = 0$.
- The equation of any plane passing through the line of intersection of the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is $(A_1x + B_1y + C_1z + D_1) + (A_2x + B_2y + C_2z + D_2) = 0$, λ being an arbitrary constant.

- The equation of any plane parallel to the plane Ax + By + Cz + D = 0 is Ax + By + Cz + k = 0, where k is any constant.



7.5 FURTHER READING

1) Jain, P. K. and Ahmhad, K. *Textbook of Analytical Geometry of Three Dimensions*. New Delhi: New Age International (P) Ltd.
2) Khan, R. M. *Analytical Geometry Of Two & Three Dimensions & Vector Analysis*. New Central Book Agency (P) Limited.
3) Kishan, Hari. *Coordinate Geometry of Two dimensions*. New Delhi: Atlantic Publishers & Distributors.
4) Pal, Sat and Pal, Harbans. *A Textbook of Two Dimensional Geometry*. New Age International Publishers.
5) Sharma, G. C. and Jain, Madhu. *Coordinate Geometry (2D and 3D)*. New Delhi: Galgotia Publications Pvt Ltd.
6) Sharma, A. K. *Coordinate Geometry (2D)*. New Delhi: Discovery Publishing House.
7) Vasistha, A. R., Agrawal, D. C. *Analytical Geometry of Three Dimensions (Coordinate of Solid Geometry)*.



7.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: Let the coordinates of A, B, C be (a, 0, 0), (0, b, 0), (0, 0, c) respectively. Then the centroid of the triangle ABC is,

$$\left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right)$$

i.e., $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$.

But the centroid of the triangle ABC is given to be (p, q, r).

So we get, $\frac{a}{3} = p$, $\frac{b}{3} = q$, $\frac{c}{3} = r$ and hence $a = 3p$, $b = 3q$, $c = 3r$.

$\therefore$ The required equation of the plane is,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

i.e., $\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}=1$

i.e., $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$.

Ans. to Q. No. 2: The equation of the plane passing through the points (1, 0, –1), (2, –3, 4) and (3, 1, 1) is,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

i.e., $\begin{vmatrix} x-1 & y-0 & z+1 \\ 2-1 & -3-0 & 4+1 \\ 3-1 & 1-0 & 1+1 \end{vmatrix}=0$

i.e., $\begin{vmatrix} x-1 & y & z+1 \\ 1 & -3 & 5 \\ 2 & 1 & 2 \end{vmatrix}=0$

i.e., $(x-1)(-6-5)-y(2-10)+(z+1)(1+6)=0$

i.e., $-11x+11+8y+7z=0$

i.e., $11x-8y-7z=18$

We express the equation in the intercept form:

Dividing both sides by 18, we get,

$$\frac{11x}{18}-\frac{8y}{18}-\frac{7z}{18}=1$$

i.e., $\frac{x}{\frac{18}{11}}+\frac{y}{-\frac{9}{4}}+\frac{z}{-\frac{18}{7}}=1$

$\therefore$ Intercepts made by the line on the coordinate axes are,

$\frac{18}{11}, -\frac{9}{4}, -\frac{18}{7}$.

Ans. to Q. No. 3: As the required plane passes through the x-axis, its equation is of the form by + cz = 0 ... ...(i).

As it has to pass through the point (1, –4, 2), we must have,

$–4b + 2c = 0 \Rightarrow 2c = 4b \Rightarrow c = 2b$.

putting for c in (i), we get, by + 2bz = 0, i.e., y + 2z = 0 as the required plane.

Ans. to Q. No. 4: We know that the two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are orthogonal if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Here for the two given planes

$x + 2y + 2z – 5 = 0$ and $2x + y – 2z = 0$.

We have, $a_1a_2 + b_1b_2 + c_1c_2 = 1.2 + 2.1 + 2.(–2) = 2 + 2 – 4 = 0$;

Hence the given two planes are orthogonal (i.e.perpendicular).

Any plane through the point (2, –4, 7) is given by

$A(x – 2) + B(y + 4) + C(z – 7) = 0$ ... ...(i)

If it is perpendicular to both the given planes, then

$A.1 + B.2 + C.2 = 0$, i.e., $A + 2B + 2C = 0$ ... ...(ii)

and $A.2 + B.1 + C.(–2) = 0$, i.e., $2A + B – 2C = 0$ ... ...(iii)

Eliminating A, B, C from (i), (ii) and (iii), we get,

$$\begin{vmatrix} x-2 & y+4 & z-7 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{vmatrix} = 0$$

i.e., $(x – 2)(–4 – 2) – (y + 4)(–2 – 4) + (z – 7)(1 – 4) = 0$

i.e., $–6x + 12 + 6y + 24 – 3z + 21 = 0$

i.e., $–6x + 6y – 3z = –57$

i.e., $2x – 2y + z = 19$

This is the required plane.

Direction ratios of the normal to the last plane are 2, –2, 1; as

$\sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = \sqrt{3}$,

the direction cosines of the said normal are, $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$.

Ans. to Q. No. 5: The given plane is $3x + 2y – 6z + 14 = 0$;

i.e., $3x + 2y – 6z = –14$ (constant term taken on the R.H.S)

In this form the coefficients of x, y, z are –3, –2, 6; dividing both sides by $\sqrt{(-3)^2 + (-2)^2 + 6^2}$, i.e., $\sqrt{9 + 4 + 36}$, i.e., 7, we get,

$\left(-\frac{3}{7}\right)x+\left(-\frac{2}{7}\right)y+\frac{6}{7}z=2$.

This is the required normal form of equation of the given plane.

Ans. to Q. No. 6: Any plane passing through the line of intersection of the planes x – y + 2z = 0 and 2x + 3y – z + 2 = 0 is represented by

x – y + 2z – 3 + k(2x + 3y – z + 2) = 0 ... ...(i)

If it passes through the point (2, 0, –5), we get,

2 – 0 – 10 – 3 + k(4 + 0 + 5 + 2) = 0

$\Rightarrow$ 11k = 11

$\Rightarrow$ k = 1

with this value of k, (i) becomes,

x – y + 2z – 3 + 2x + 3y – z + 2 = 0

i.e., 3x + 2y + z –1 = 0

Which is the required equation of the proposed plane.

Ans. to Q. No. 7: If θ be the angle between the planes x – 2y + 2z – 5 = 0 and 3x – 2y + 6z = 0, then,

$$\cos\theta = \frac{1.3+(-2).(-2)+2.6}{\sqrt{1^2+(-2)^2+2^2}\sqrt{3^2+(-2)^2+6^2}}$$

$$= \frac{3+4+12}{\sqrt{9}\sqrt{49}}$$

$$= \frac{19}{21}$$

$\Rightarrow \theta = \cos^{-1}\frac{19}{21}$.

Ans. to Q. No. 8: In order that the plane kx – y + 4z = 2 and 3x + 2ky – 11z = 8 may be orthogonal (i.e., perpendicular), we must have k.3 + (–1).2k + 4.(–11) = 0 [condition of perpendicularity] $\Rightarrow$ k = 44.



7.7 MODEL QUESTIONS

Q.1: Find the equation of the plane which passes through the point (3, –3, 1) and is normal to the line joining (3, 2, –1) and (2, –1, 5).

Q.2: Find the angle between the planes:

i) $2x + 3y – 5z = 6, 3x + 8y + 6z = 9$

ii) $2x – y + 2z – 6 = 0, 3x + 6y + 2z – 9 = 0$

Q.3: Find the equation of the plane through (1, 2, 3) parallel to the plane $4x + 5y + –3z = 7$.

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UNIT 8: THE STRAIGHT LINE

UNIT STUCTURE

8.1 Learning Objectives

8.2 Introduction

8.3 The Straight Line

8.4 The Shortest Distance (S.D) Between Two Lines

8.5 Let Us Sum Up

8.6 Further Reading

8.7 Answers to Check Your Progress

8.8 Model Questions

8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know about the plane and the straight line
- find shortest distance(S.D) between two lines.

8.2 INTRODUCTION

In unit 7, we have discussed the equation of plane. In this unit, we will discuss the equation of line in space. Here, we will also find the shortest distance between two lines.

8.3 THE STRAIGHT LINE

a) The Equation of a Line:

A pair of first degree equations taken together,

$a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$

represent a straight line in space, because the coordinates of any point on the line of intersection of the two planes represented by them satisfy both the equations; it is understood that the two planes are not parallel. Thus, $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ represent a straight line, viz. the line of intersection of the two planes represented by the two equations separately.

b) The symmetrical form of the equations of a straight line:

Suppose that a line passes through a given point $A(x_1, y_1, z_1)$ and has direction ratios l, m, n.

Let P(x, y, z) be any point on the line. As the line passes through the points $A(x_1, y_1, z_1)$ and P(x, y, z), it has direction ratios $x - x_1$, $y - y_1$, $z - z_1$. Thus, $x - x_1$, $y - y_1$, $z - z_1$ and l, m, n are two sets of direction ratios of the same line and so we have,

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

But P(x, y, z) is any point on the line. So it follows that the line is represented by the equations,

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n},$$

known as the symmetrical form of the equations of the line.

Remarks:

i) In particular l, m, n may be the direction cosines of the line.

ii) The equations $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = k$, enable us to express the coordinates of variable point on the line in terms of a parameter k, for $x = x_1 + lk$, $y = y_1 + mk$, $z = z_1 + nk$.

iii) For this line we have,

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = k^2(l^2 + m^2 + n^2)$$

$$\Rightarrow k^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2,$$

in case l, m, n are the direction cosines of the line.It follows that the numerical value of k (i.e., |k|) is the distance between the variable point.

P(x, y, z) and the fixed point (x_1, y_1, z_1) on the line. k, which may be positive or negative, is referred to as the algebraic distance of a variable point on the line from the fixed point (x_1, y_1, z_1).

In case l, m, n are not actual direction cosines of the line, k is proportional to the algebraic distance of the variable point (x, y, z) from the point (x_1, y_1, z_1).

c) Line through two points:

If a line passes through the two points (x_1, y_1, z_1) and (x_2, y_2, z_2), then it has direction ratios $x_2 - x_1, y_2 - y_1, z_2 - z_1$. Hence the equations of the line are,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

ILLUSTRATIVE EXAMPLES:

Example 1: Find the equations of the line passing through the point (1, –4, 3) and parallel to the line joining the points (–1, 5, 2) and (2, –1, 0).

Solution: Direction ratios of the line joining the points (–1, 5, 2) and (2, –1, 0) are, 2 – (–1), –1 – 5, 0 – 2, i.e., 3, –6, –2.

The required line is parallel to the above and so has direction ratios 3, –6, –2; also it passes through the point (1, –4, 3).

Hence the equations to the required line are,

$$\frac{x-1}{3} = \frac{y-(-4)}{-6} = \frac{z-3}{-2}$$

i.e., $\frac{x-1}{3} = \frac{y+4}{-6} = \frac{z-3}{-2}$

Example 2: Find the equations of the line passing through the points (–1, 2, –3) and (2, 1, –4). Where does it meet the plane x – 3y + 4z – 5 = 0?

Solution: The equations of the line passing through the points (–1, 2, –3) and (2, 1, –4) are,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

i.e., $\frac{x-(-1)}{2-(-1)} = \frac{y-2}{1-2} = \frac{z-(-3)}{-4-(-3)}$

i.e., $\frac{x+1}{3} = \frac{y-2}{-1} = \frac{z+3}{-1}$

The line is given by $\frac{x+1}{3} = \frac{y-2}{-1} = \frac{z+3}{-1} = k$ (say)

then $x + 1 = 3k, y - 2 = -k, z + 3 = -k$

$\Rightarrow x = 3k - 1, y = 2 - k, z = -k - 3$

$\therefore$ Any point on the above line is $(3k - 1, 2 - k, -k - 3)$.

If this point lies on the plane x – 3y + 4z – 5 = 0**,** we have,

$(3k - 1) - (2 - k) + (-k - 3) - 5 = 0$

$\Rightarrow 3k - 1 - 6 + 3k - 4k - 12 - 5 = 0$

$\Rightarrow 2k - 24 = 0$

$\Rightarrow 2k = 24$

$\Rightarrow k = 12$

For k = 12, we have,

$x = 3k - 1 = 35,\ y = 2 - k = -10,\ z = -k - 3 = -15$

$\therefore$ The line meets the given plane at the point (35, –10, –15).

Example 3: Find the direction cosines of the line represented by the equations $6x + 2y - 3z - 14 = 0 = x + 2y - z + 8$

Solution: The given line is the line of intersection of the two planes

$$\left.\begin{aligned} 6x + 2y - 3z - 14 = 0 \\ x + 2y - z + 8 = 0 \end{aligned}\right\} \quad \text{... ... ...(1)}$$

The equation of the plane through the origin parallel to the plane $6x + 2y - 3z - 14 = 0$ is $6x + 2y - 3z = 0$ and that of the plane through the origin parallel to the plane $x + 2y - z + 8 = 0$ is $x + 2y - z = 0$. Then the pair of equations,

$$\left.\begin{aligned} 6x + 2y - 3z = 0 \\ x + 2y - z = 0 \end{aligned}\right\} \quad \text{... ... ...(2)}$$

together represent the line through the origin parallel to the given line (1). Now (2) may be expressed as,

$$\frac{x}{-2+6} = \frac{y}{-3+6} = \frac{z}{12-2}$$

i.e., $\frac{x}{4} = \frac{y}{3} = \frac{z}{10}$

Which are the equations of the line (2) in the symmetrical form. So the line has direction ratios 4, 3, 10; the given line (1) being parallel to the line (2) has direction ratios 4, 3, 10.

As $\sqrt{4^2 + 3^2 + 10^2} = \sqrt{125} = 5\sqrt{5}$.

the direction cosines of the given line (1) are, $\frac{4}{5\sqrt{5}}, \frac{3}{5\sqrt{5}}, \frac{10}{5\sqrt{5}}$.

Example 4: Find the equations of the line $2x + 3y - z = 0 = 4x + y - 5z + 12$ in the symmetrical.

Solution: The equations of the line are

$2x + 3y – z = 0 = 4x + y – 5z + 12$ ... ...(1)

The equations of the line through the origin parallel to the line are,

$$\left.\begin{aligned} 2x + 3y - z = 0 \\ 4x + y - 5z = 0 \end{aligned}\right\}$$ ... ...(2)

Which may be expressed as,

$$\frac{x}{-15+1} = \frac{y}{-4+10} = \frac{z}{2-12}$$

i.e., $\frac{x}{-14} = \frac{y}{6} = \frac{z}{-10}$

i.e., $\frac{x}{7} = \frac{y}{-3} = \frac{z}{5}$

It follows that 7, –3, 5 are direction ratios of the line (2) and hence of the given line (1) as these two lines are parallel.

Next we need a point on the given line (1). Putting z = 0 in (1),

we get, $\left.\begin{aligned} 2x + 3y - 6 = 0 \\ 4x + y + 12 = 0 \end{aligned}\right\}$

Thus the line (1) passes through the point $\left(-\frac{21}{5}, \frac{24}{5}, 0\right)$; also it has direction ratios 7, –3, 5.

Hence the equation of the given line (1) in symmetrical form an

$$\frac{x + \frac{21}{5}}{7} = \frac{y - \frac{24}{5}}{-3} = \frac{z}{5}.$$

Example 5: Find the angle between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$ and $2x + 2y – z = 0 = 2x +3 – 4z – 8$.

Solution: The given lines are,

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$$ ... ...(1)

and $2x + 2y – z – 6 = 2x + 3 – 4z – 8$ ... ...(2)

Let l, m, n be the direction cosines of the line (2). Then,

$$\left.\begin{aligned} 2l + 2m - n = 0 \\ 2l + 3m - 4n = 0 \end{aligned}\right\}$$

[This is because the line (2) lies in the planes $2x + 2y - z - 6 = 0$ and $2x + 3 - 4z - 8 = 0$ and hence the normals to the two planes having direction ratios 2, 2, –1 and 2, 3, –4 are perpendicular to the line.]

We get, $\frac{l}{-8+3} = \frac{m}{-2+8} = \frac{n}{6-4} \Rightarrow \frac{l}{-5} = \frac{m}{6} = \frac{n}{2}$

$\therefore$ –5, 6, 2 are direction ratios of line (2).

Thus line (1) has direction ratios 2, 3, 4 and line (2) has direction ratios –5, 6, 2. Hence the required angle between the two lines

$$= \cos^{-1} \frac{2.(-5) + 3.6 + 4.2}{\sqrt{2^2 + 3^2 + 4^2}\sqrt{(-5)^2 + 6^2 + 2^2}}$$

$$= \cos^{-1} \frac{16}{\sqrt{29}\sqrt{65}}$$



CHECK YOUR PROGRESS

Q.1: Find the equations of the line through the point (α, β, γ) parallel to the y-axis.

Q.2: Prove that the equations of the line $3x + 2y - 3z + 3 = 0 = x + 2y - 2z + 4$ can be put in the form $\frac{x-2}{2} = \frac{y}{3} = \frac{z-3}{4}$.

Q.3: A line through the point (0, 3, –1) is perpendicular to each of the lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and $\frac{x-1}{4} = \frac{y-2}{5} = \frac{z+1}{-1}$.

Find the equations of the line.

Q.4: Find the ratio in which the join of the points A(2, 3, 1) and B(–2, 1, –3) is divided by the plane $x - 2y + 3z + 4 = 0$. Find also the coordinates of the point of intersection.

Q.5: Find the angle between the line $\frac{x+1}{2} = y - 3 = \frac{z-2}{-2}$ and the plane $3x - 4z + 8 = 0$.

Q.6: Find the value of λ so that the lines $\frac{x+7}{\lambda} = \frac{y-9}{1} = \frac{z+1}{-2}$ and $\frac{x+4}{5} = \frac{y+3}{-3\lambda} = \frac{z-3}{2}$ are perpendicular.

A plane and a straight line:

Consider the plane $Ax + By + Cz + D = 0$ ... ...(1)

and the line $\frac{x - x_1`}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ ... ...(2)

Any pont of the line (2) is $(x_1 + lk, y_1 + mk, z_1 + nk)$, k being a parameter; this line will intersect the plane (1), if for some value of k, this point lies in the plane (1), i.e. if,

$A(x_1 + lk) + B(y_1 + mk) + C(z_1 + nk) + D = 0$ ... ...(3)

i.e., $(Al + Bm + Cn)k = -(Ax_1 + By_1 + Cz_1 + D)$

i) If $Al + Bm + Cn \neq 0$, the line (2) will intersect the plane (1) at the point $(x_1 + lk, y_1 + mk, z_1 + nk)$, where, $k = -\frac{(Ax_1 + By_1 + Cz_1 + D)}{(Al + Bm + Cn)}$

ii) If $Al + Bm + Cn = 0$ and $Ax + By + Cz + D \neq 0$, then no finite value of k satisfies (3); this implies that the line (2) does not have any point in common with the plane (1), i.e., the line (2) does not intersect the plane (1). In other words the line (2) is parallel to the plane (1) if, $Al + Bm + Cn = 0$ and $Ax + By + Cz + D \neq 0$.

iii) If $Al + Bm + Cn = 0$ and $Ax_1 + By_1 + Cz_1 + D = 0$, then (3) is satisfied by every value of k, so that every point of the line (2) lies in the plane (1), i.e., the line (2) wholly lies in the plane (1). Thus the line (2) lies in the plane (1) if, $Al + Bm + Cn = 0$ and $Ax_1 + By_1 + Cz_1 + D = 0$.

iv) The line (2) will be a normal to the plane (1) if, $\frac{l}{A} = \frac{m}{B} = \frac{n}{C}$, because l, m, n are direction ratios of the line and A, B, C are direction ratios of normal to the plane.

v) The general equation of a plane passing through the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is, $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$, Where, $Al + Bm + Cn = 0$.

ILLUSTRATIVE EXAMPLES:

Example 1: Prove that the line $\frac{x - 3}{2} = \frac{y - 4}{3} = \frac{z - 5}{4}$, (i) is parallel to the plane $4x + 4y - 5z + 3 = 0$ and (ii) lies wholly in the plane $4x + 4y - 5z - 3 = 0$.

Solution: i) In relation to the line $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}$ and the plane $4x + 4y – 5z – 3 = 0$, we have, in usual notation,

$Al + Bm + Cn = 4.2 + 4.3 + (–5).4 = 8 + 12 – 20 = 0$

and $Ax_1 + By_1 + Cz_1 + D = 4.3 + 4.4 – 5.5 + 3 – 31 – 25 \neq 0$;

it follows that the given line is parallel to the plane $4x + 4y – 5z + 3 = 0$.

ii) Next consider the given line $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}$ and the plane $4x + 4y – 5z – 3 = 0$; have in usual notations,

$Al + Bm + Cn = 4.2 + 4.3 + (–5).4 = 8 + 12 – 20 = 0$

and $Ax_1 + By_1 + Cz_1 + D = 4.3 + 4.4 – 5.5 – 3 = 28 – 28 = 0$;

It follows that the line $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}$ lies wholly in the plane $3x + 4y – 5z – 3 = 0$.

Example 2: Show that the equation to the plane through the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and parallel to the line $\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}$ is, $\sum(x – \alpha)(mn' – m'n) = 0$.

Solution: The equation of any plane through the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is,

$$A(x – \alpha) + B(y – \beta) + C(z – \gamma) \quad \text{... ... ...(1)}$$

Where, $Al + Bm + Cn = 0$ ... ...(2)

If this plane is parallel to the line $\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}$, then we must have,

$Al' + Bm' + Cn' = 0$ ... ...(3)

Eliminating A, B, C from (1), (2), (3) we get the equation to the required plane as,

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$$

i.e., $(x – \alpha)(mn' – m'n) + (y – \beta)(nl' – n'l) + (z – \gamma)(lm' – l'm) = 0$

i.e., $\sum(x – \alpha)(mn' – m'n) = 0$.

Example 3: Find the distance of the point (2, –1, 3) from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane x – y + z = 5.

Solution: $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ ... ...(i)

x – y + z = 5 ... ...(ii)

In (i) putting each ratio = ρ, we get,

$x - 2 = 3\rho$, $y + 1 = 4\rho$, $z - 2 = 12\rho$

$\Rightarrow$ $x = 3\rho + 2$, $y\ 4\rho - 1$, $z = 12\rho + 2$ ... ...(iii)

Putting these values of x, y, z in (ii), we get,

$(3\rho + 2) - (4\rho - 1) + (12\rho + 2) = 5$

$\Rightarrow$ $11\rho + 5 = 5$

$\Rightarrow$ $11\rho = 0$

$\Rightarrow$ $\rho = 0$

Putting $\rho = 0$ in (iii), x = 2, y = –1, z = 2.

So the point of intersection of the line (i) and the plane (ii) has coordinates (2, –1, 2).

$\therefore$ Required distance of the point (2, –1, 3) from this point of intersection,

(2, –1, 2) = $\sqrt{(2-2)^2+(-1+1)^2+(3-2)^2}$ = $\sqrt{0+0+1}=\sqrt{1}=1$.

Example 4: Find the equation of the plane through the line 2x – y + 3z + 1 = 0 = x + y + z + 3 parallel to the line $\frac{x+1}{1}=\frac{y-5}{2}=\frac{z}{3}$.

Solution: Any plane through the line

2x – y + 3z + 1 = 0 = x + y + z + 3 is,

$2x - y + 3z + 1 + \lambda(x + y + z + 3) = 0$ ... ...(1)

It will be parallel to the line

$\frac{x+1}{1}=\frac{y-5}{2}=\frac{z}{3}$ if the condition Al + Bm + Cn = 0 is satisfied

i.e., if $(2 + \lambda).1 + (\lambda - 1).2 + (3 + \lambda).3 = 0$

$\Rightarrow$ $2 + \lambda + 2\lambda - 2 + 9 + 3\lambda = 0$

$\Rightarrow$ $6\lambda = -9$

$\Rightarrow$ $\lambda = -\frac{3}{2}$

Putting $\lambda = -\frac{3}{2}$ in (1) we get,

$$\left(2-\frac{3}{2}\right)x+\left(-\frac{3}{2}-1\right)y+\left(3-\frac{3}{2}\right)z+\left(1-3.\frac{3}{2}\right)=0$$

i.e., $\frac{1}{2}x-\frac{5}{2}y+\frac{3}{2}z-\frac{7}{2}=0$

i.e., $x - 5y + 3z - 7 = 0$

as the required equation of the desired plane.

Coplanar Lines: Two lines are coplanar if they lie in the same plane; otherwise they are non-coplanar or skew lines. Two coplanar lines either intersect at a point or are parallel.

Consider the two lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \qquad \text{... ... ...(i)}$$

and $$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \qquad \text{... ... ...(ii)}$$

The equation to a plane through the line (i) is,

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \qquad \text{... ... ...(iii)}$$

Where, $Al_1 + Bm_1 + Cn_1 = 0$ ... ...(iv)

If this plane contains line (ii), then,

$$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0 \qquad \text{... ... ...(v)}$$

and $Al_2 + Bm_2 + Cn_2 = 0$ ... ...(vi)

Eliminating A,B,C from (iv), (v) and (vi), we get,

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \qquad \text{... ... ...(vii)}$$

Which is the required condition for the lines (i) and (ii) to be coplanar. When this condition is satisfied, the lines (i) and (ii) are coplanar, and the equation of the plane containing them is obtained by eliminating A, B, C from (iii), (iv), (vi) as,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

Example 1: Prove that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ are coplanar and find the plane containing them.

Solution: The given lines are, $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$

Now, $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$

$= \begin{vmatrix} 1-2 & 2-3 & 3-4 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$= \begin{vmatrix} -1 & -1 & -1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$= \begin{vmatrix} -1 & 0 & 0 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$ (by $C_2 - C_3$ and $C_3 - C_2$)

$= 0$

$\therefore$ the lines are coplanar.

8.4 THE SHORTEST DISTANCE (S.D) BETWEEN TWO LINES

The line of shortest distance between two given straight lines is the straight line which is perpendicular to both the given lines; the length of the line segment intercepted by the two lines on their common perpendicular (i.e., on their line of shortest distance) is called the shortest distance (S.D) between the two lines. The following results holds:

The shortest distance (S.D) between two lines:

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$

is given by S.D = $\frac{|\Delta|}{k}$

Where, $\Delta = \begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$

and $k = \sqrt{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2+\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2+\begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2}$

In case $\Delta = 0$, we get S.D = 0 which implies that two lines are coplanar and intersecting.

Distance of a given point from a given line:

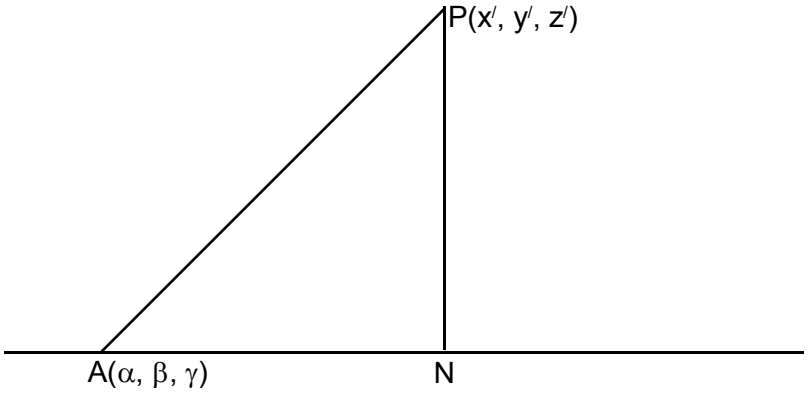


Fig. 8.1

If a line passes through a point A(α, β, γ) and has l, m, n as its direction cosines, then the length of the perpendicular PN, drawn from the point P(x', y', z') to the line, is given by

$$PM^2 = \begin{vmatrix} x'-\alpha & y'-\beta \\ l & m \end{vmatrix}^2+\begin{vmatrix} y'-\beta & z'-\gamma \\ m & n \end{vmatrix}^2+\begin{vmatrix} z'-\gamma & x'-\alpha \\ n & l \end{vmatrix}^2$$

Example 1: Find the S.D between the lines

$$\frac{x-1}{2}=\frac{y+8}{-7}=\frac{z-4}{5} \text{ and } \frac{x-1}{2}=\frac{y-2}{1}=\frac{z-6}{-3}$$

Solution: We have, $\Delta = \begin{vmatrix} 1-1 & -8-2 & 4-6 \\ 2 & -7 & 5 \\ 2 & 1 & -3 \end{vmatrix}$

$$= \begin{vmatrix} 0 & -10 & -2 \\ 2 & -7 & 5 \\ 2 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 0 & -10 & -2 \\ 0 & -8 & 8 \\ 2 & 1 & -3 \end{vmatrix}$$

$$= 2 \times (-80 - 16)$$

$$= -192$$

and $k = \sqrt{\begin{vmatrix} 2 & -7 \\ 2 & 1 \end{vmatrix}^2 + \begin{vmatrix} -7 & 5 \\ 1 & -3 \end{vmatrix}^2 + \begin{vmatrix} 5 & 2 \\ -3 & 2 \end{vmatrix}^2}$

$$= \sqrt{(2+14)^2 + (21-5)^2 + (10+6)^2}$$

$$= \sqrt{3 \times 16^2}$$

$$= 16\sqrt{3}$$

$\therefore$ Required S.D $= \dfrac{|\Delta|}{k}$

$$= \frac{192}{16\sqrt{3}}$$

$$= \frac{12}{\sqrt{3}}$$

$$= \frac{12\sqrt{3}}{3}$$

$$= 4\sqrt{3}$$

Example 2: Calculate the distance of the point (2, 3, 4) from the line $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-3}{3}$.

Solution: The line passes through the point $(\alpha, \beta, \gamma) = (1, 2, 3)$ and has direction the ratios 1, 2, 3; its direction cosines l, m, n are given by

$$l = \frac{1}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{14}}, m = \frac{2}{\sqrt{14}}, n = \frac{3}{\sqrt{14}}$$

If d be the the required distance,

$$d^2 = \begin{vmatrix} x'-\alpha & y'-\beta \\ l & m \end{vmatrix}^2 + \begin{vmatrix} y'-\beta & z'-\gamma \\ m & n \end{vmatrix}^2 + \begin{vmatrix} z'-\gamma & x'-\alpha \\ n & l \end{vmatrix}^2$$

$$= \begin{vmatrix} 2-1 & 3-2 \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{vmatrix}^2 + \begin{vmatrix} 3-2 & 4-3 \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{vmatrix}^2 + \begin{vmatrix} 4-3 & 2-1 \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{vmatrix}^2$$

$$= \left(\frac{2}{\sqrt{14}} - \frac{1}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}} - \frac{2}{\sqrt{14}}\right)^2 + \left(\frac{1}{\sqrt{14}} - \frac{3}{\sqrt{14}}\right)^2$$

$$= \frac{1}{14} + \frac{1}{14} + \frac{4}{14}$$

$$= \frac{6}{14}$$

$$= \frac{3}{7}$$

$\therefore d = \sqrt{\frac{3}{7}}$, which is the required distance.



CHECK YOUR PROGRESS

Q.7: Find the plane containing the point (1, 4, –3) and the line given by $3x + 2y - 2 = 0$ and $2y + z + 3 = 0$.

Q.8: Show that the equation of the plane through the line $x + y - 2z + 4 = 0 = 3x - y + 2z + 1$ and parallel to the line $\frac{x+2}{2} = \frac{y}{3} = \frac{z-4}{-1}$ is $2x - 8y + 16z + 3 = 0$.

Q.9: Find the equation of the plane through the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{4}$ and perpendicular to the plane $3x - y + 4 = 0$.

Q.10: Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $4x - 3y + 1 = 0 = 5x - 3z + 2$ are coplanar.

Q.11: Show that the lines $\frac{x+1}{3}=\frac{y-1}{3}=\frac{z-2}{-2}$ and $\frac{x-3}{1}=\frac{y-6}{2}=\frac{z+3}{-3}$ are coplanar and find their point of intersection.

Q.12: Find the S.D between the z-axis and the line $3x - 4y + 5 = 0 = 6x - 2y + z + 9$.

Q.13: Find the S.D between the lines $\frac{x-1}{1}=\frac{y-1}{1}=\frac{z-1}{3}$ and $\frac{x-4}{2}=\frac{y-6}{3}=\frac{z-8}{4}$. How do you interpret the result?

Q.14: Find the distance of the point (1, 2, 3) from the line $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-5}{4}$.

Q.15: Find the condition that the lines $x = az + b$, $y = cz + d$ and $x = a'z + b'$, $y = c'z + d'$ may be coplanar.



8.5 LET US SUM UP

- A pair of first degree equations in x, y, z taken together represents a straight line in space.
- The equation to the line passing through the point (x_1, y_1, z_1) and having direction ratios l, m, n are $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$.
- The equations to the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$.
- The line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$
  i) is parallel to the plane $Ax + By + Cz + D = 0$ if $Al + Bm + Cn = 0$ and $Ax + By + Cz + D \neq 0$;
  ii) lies in the plane $Ax + By + Cz + D = 0$ if $Al + Bm + Cn = 0$ and $Ax_1 + By_1 + Cz_1 + D = 0$;

iii) is perpendicular (or normal) to the plane Ax + By + Cz + D = 0 if $\frac{l}{A} = \frac{m}{B} = \frac{n}{C}$.

- The equation of any plane containing the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$, where $Al + Bm + Cn = 0$.
- The lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ will be coplanar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$ and in that event the equation of the plane containing the two lines is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.
- The shortest distance (S.D) between the two lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ is given by the formula S.D = $\frac{|\Delta|}{k}$

  Where $\Delta = \begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$

  and $k = \sqrt{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2 + \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2}$
- The (perpendicular) distance, d, of a given point (x', y', z') from the line $\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$ $(l^2 + m^2 + n^2 = 1)$

  $d^2 = \begin{vmatrix} x' - \alpha & y' - \beta \\ l & m \end{vmatrix}^2 + \begin{vmatrix} y' - \beta & z' - \gamma \\ m & n \end{vmatrix}^2 + \begin{vmatrix} z' - \gamma & x' - \alpha \\ n & l \end{vmatrix}^2$



8.6 FURTHER READING

1) Jain, P. K. and Ahmhad, K. *Textbook of Analytical Geometry of Three Dimensions*. New Delhi: New Age International (P) Ltd.
2) Khan, R. M. *Analytical Geometry Of Two & Three Dimensions & Vector Analysis*. New Central Book Agency (P) Limited.
3) Kishan, Hari. *Coordinate Geometry of Two dimensions*. New Delhi: Atlantic Publishers & Distributors.
4) Pal, Sat and Pal, Harbans. *A Textbook of Two Dimensional Geometry*. New Age International Publishers.
5) Sharma, G. C. and Jain, Madhu. *Coordinate Geometry (2D and 3D)*. New Delhi: Galgotia Publications Pvt Ltd.
6) Sharma, A. K. *Coordinate Geometry (2D)*. New Delhi: Discovery Publishing House.
7) Vasistha, A. R., Agrawal, D. C. *Analytical Geometry of Three Dimensions (Coordinate of Solid Geometry)*.



8.7 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: The direction cosines of y-axis are 0, 1, 0. The required line is parallel to y-axis and so its direction cosines are 0, 1, 0; also, it passes through the point (α, β, γ). Hence the equations of the line are, $\frac{x-\alpha}{0} = \frac{y-\beta}{1} = \frac{z-\gamma}{0}$

i.e., $x = \alpha$, $z = \gamma$.

Ans. to Q. No. 2: The given line is represented by the equations,

$$\left.\begin{aligned} 3x+2y-3z+3=0 \\ x+2y-2z+4=0 \end{aligned}\right\} \quad \text{... ... ...(1)}$$

Suppressing the constants in (1), we get,

$$\left.\begin{aligned} 3x+2y-3z=0 \\ x+2y-2z=0 \end{aligned}\right\} \quad \text{... ... ...(2)}$$

Which represent a line through the origin parallel to line (1); (2) can be expressed as, $\frac{x}{-4+6} = \frac{y}{-3+6} = \frac{z}{6-2}$ i.e., $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$

Which are equations of line (2) in symmetric forms. we see that the line (2) has direction ratios 2, 3, 4 and hence the line (1), which is parallel to line (2), has direction ratios 2, 3, 4. We now need a point on line (1).

Putting y = 0 in (1), we get, $\left.\begin{aligned} 3x - 3z + 3 = 0 \\ x - 2z + 4 = 0 \end{aligned}\right\}$

i.e., $\left.\begin{aligned} x - z + 1 = 0 \\ x - 2z + 4 = 0 \end{aligned}\right\}$,

Whence, on subtraction, we get, $z - 3 = 0 \Rightarrow z = 3$

Hence from the first equation above, $x - 3 + 1 = 0 \Rightarrow x = 2$.

It follows that (2, 0, 3) is a point on the line (1) which has direction ratios 2, 3, 4. Hence the equations of the given line (1) can be put in the symmetrical form $\frac{x-2}{2} = \frac{y}{3} = \frac{z-3}{4}$.

Ans. to Q. No. 3: Let l, m, n be the direction cosines of the line under consideration. As it is perpendicular to each of the lines,

$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and $\frac{x-1}{4} = \frac{y-2}{5} = \frac{z+1}{-1}$

We have, $\left.\begin{aligned} l - 2m + 3n = 0 \\ 4l + 5m - n = 0 \end{aligned}\right\}$

Giving, $\frac{l}{2-15} = \frac{m}{12+1} = \frac{n}{5+8}$

$\Rightarrow \frac{l}{-13} = \frac{m}{13} = \frac{n}{13}$

$\Rightarrow \frac{l}{-1} = \frac{m}{1} = \frac{n}{1}$

$\therefore$ The line has direction ratios –1, 1, 1.

Also it is given that it passes through the point (0, 3, –1). Hence the equation of the line are, $\frac{x}{-1} = \frac{y-3}{1} = \frac{z+1}{1}$.

Ans. to Q. No. 4: Suppose that the join of the points A(2, 3, 1) and B(–2, 1, –3) intersects the plane x – 2y + 3z + 4 = 0 at P(α, β, γ).

Let AP : PB = λ : 1; then,

$$\left.\begin{aligned}\alpha &= \frac{\lambda.(-2)+1.2}{\lambda+1} = \frac{-2\lambda+2}{\lambda+1}\\ \beta &= \frac{\lambda.1+1.3}{\lambda+1} = \frac{\lambda+3}{\lambda+1}\\ \gamma &= \frac{\lambda.(-3)+1.1}{\lambda+1} = \frac{1-3\lambda}{\lambda+1}\end{aligned}\right\} \quad \text{... ... ...(i)}$$

As P(α, β, γ) lies in the plane x – 2y + 3z + 4 = 0, we get,

$$\frac{2-2\lambda}{\lambda+1} - 2\left(\frac{\lambda+3}{\lambda+1}\right) + 3\left(\frac{1-3\lambda}{\lambda+1}\right) + 4 = 0$$

$\Rightarrow$ 2 – 2λ – 2λ – 6 + 3 – 9λ + 4 = 0

$\Rightarrow$ 9λ + 3 = 0

$\Rightarrow$ 9λ = 3

$\Rightarrow$ $9\lambda = \frac{1}{3}$

$\therefore$ Putting this value of λ in (i) we get,

$$\alpha = \frac{2-2.\frac{1}{3}}{\frac{1}{3}+1} = \frac{\frac{4}{3}}{\frac{4}{3}} = 1$$

$$\beta = \frac{\frac{1}{3}+3}{\frac{1}{3}+1} = \frac{10}{3}\times\frac{3}{4} = \frac{5}{2}$$

$$\gamma = \frac{1-3.\frac{1}{3}}{\frac{1}{3}+1} = 0$$

$\therefore$ The coordinates of the point of intersection P are $\left(1, \frac{5}{2}, 0\right)$.

Ans. to Q. No. 5: Let θ be the angle between the line $\frac{x+1}{2} = y-3 = \frac{z-2}{-2}$ and the plane $3x - 4z + 8 = 0$. Then the angle between the line, whose direction ratios are 2, 1, –2 and the normal to the plane, which has direction ratios 3, 0, –4 is $90^0 - \theta$.

So we have, $\cos(90^0 - \theta) = \frac{2.3 + 1.0 + (-2).(-4)}{\sqrt{2^2 + 1^2 + (-2)^2}\sqrt{3^2 + 0^2 + (-4)^2}}$

$$\Rightarrow \sin\theta = \frac{6+8}{\sqrt{9}\sqrt{25}} = \frac{14}{15}$$

$$\Rightarrow \theta = \sin^{-1}\frac{14}{15}.$$

Ans. to Q. No. 6: In order that the lines, $\frac{x+7}{\lambda} = \frac{y-9}{1} = \frac{z+1}{-2}$ and $\frac{x+4}{5} = \frac{y+3}{-3\lambda} = \frac{z-3}{2}$ may be perpendicular, we must have (by the condition of perpendicularity $l_1l_2 + m_1m_2 + n_1n_2 = 0$).

$\lambda.5 + 1.(-3\lambda) + (-2).2 = 0$

$\Rightarrow \lambda = 4$

$\Rightarrow \lambda = 2$

Ans. to Q. No. 7: Any plane containing the line given by $3x + 2y - 2 = 0$ and $2y + z + 3 = 0$ is $3x + 2y - 2 + \lambda(2y + z + 3) = 0$ ... ...(i)

This plane will contain the point (1, 4, –3) also if,

$3.1 + 2.3 - 2 + \lambda(2.4 - 3 + 3) = 0$

$\Rightarrow 8\lambda + 9 = 0$

$\Rightarrow 8\lambda = -9$

$\Rightarrow \lambda = -\frac{9}{8}$

Hence (i) becomes, $3x + 2y - 2 - \frac{9}{8}(2y + z + 3) = 0$

i.e., $24x + 16y - 16 - 18y - 9z - 27 = 0$

i.e., $24x - 2y - 9z - 43 = 0$

Which is the equation of the required plane.

Ans. to Q. No. 8: The equation of any plane through the line, $x + y - 2z + 4 = 0 = 3x - y + 2z + 1$ is,

$x + y - 2z + 4 + \lambda(3x - y + 2z + 1) = 0$

i.e., $(1 + 3\lambda)x + (1 - \lambda)y + (2\lambda - 2)z + (\lambda + 4) = 0$ ... ...(i)

This plane will be parallel to the line $\frac{x+2}{2} = \frac{y}{3} = \frac{z-4}{-1}$ if the condition

$Al + Bm + Cn = 0$ is satisfied.

i.e., if, $(1 + 3\lambda).2 + (1 - \lambda).3 + (2\lambda - 2).(-1) = 0$

i.e., if, $2 + 6\lambda + 3 - 3\lambda - 2\lambda + 2 = 0$

i.e., if, $\lambda = -7$.

Putting $\lambda = -7$ in (i) we get,

$(1 - 21)x + (1 + 7)y + (-14 - 2)z + (-7 + 4) = 0$

i.e., $-20x + 8y - 16z - 3 = 0$

i.e., $20x - 8y + 16z + 3 = 0$

as the required equation of the plane.

Ans. to Q. No. 9: Equation of any plane through the line

$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{4}$ is,

$A(x + 1) + B(y - 3) + C(z - 1) = 0$ ... ...(i)

Where, $2A + 3B + 4C = 0$ ... ...(ii)

If the plane (i) is perpendicular to the plane $3x - y + 4 = 0$,

we must have, $3A - B + 0.C = 0$ ... ...(iii)

Eliminating A, B, C from (i), (ii) and (iii), we get,

$$\begin{vmatrix} x+1 & y-3 & z-1 \\ 2 & 3 & 4 \\ 3 & -1 & 0 \end{vmatrix} = 0$$

i.e., $(x + 1)(0 + 4) - (y - 3)(0 - 12) + (z - 1)(-2 - 9) = 0$

i.e., $4(x + 1) + 12(y - 3) - 11(z - 1) = 0$

i.e., $4x + 4 + 12y - 36 - 11z + 11 = 0$

i.e., $3x + 12y - 11z = 21$

This is the required equation of the plane.

Ans. to Q. No. 10: The given lines are, $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ ... ...(i)

and $4x - 3y + 1 = 0 = 5x - 3z + 2$ ... ...(ii)

Any plane containing the line (ii) is,

$4x - 3y + 1 + k(5x - 3z + 2) = 0$

i.e., $(4 + 5k)x - 3y - 3kz + (1 + 2k) = 0$ ... ...(iii)

For some value of k the plane (iii) contains the line (i), the lines (i) and (ii) will be coplanar.

Now for the plane (iii) to contain (i), we must have,

$Al + Bm + Cn = 0$ and $Ax_1 + By_1 + Cz_1 + D = 0$ (in usual notation)

i.e., $(4 + 5k).2 + (-3).3 + (-3k).4 = 0$ ... ...(iv)

and $(4 + 5k).1 - 3.2 - 3k.3 + (1 + 2k) = 0$ ... ...(v)

(iv) $\Rightarrow 8 + 10k - 9 - 12k = 0 \Rightarrow -2k = 1 \Rightarrow k = -\frac{1}{2}$

(v) $\Rightarrow 4 + 5k - 6 - 9k + 1 + 2k = 0 \Rightarrow -2k = 1 \Rightarrow k = -\frac{1}{2}$

Thus for $k = -\frac{1}{2}$ both the conditions (iv) and (v) are satisfied and hence for $k = -\frac{1}{2}$ the plane (iii) contains both the lines (i) and (ii).

For $k = -\frac{1}{2}$ (iii) becomes,

$$\left(4 - \frac{5}{2}\right)x - 3y - 3.\left(-\frac{1}{2}\right)z + \left(1 - 2.\frac{1}{2}\right) = 0$$

i.e., $\frac{3x}{2} - 2y + \frac{3}{2}z = 0$

i.e., $x - 2y + z = 0$

Hence the given lines (i) and (ii) are coplanar as they both lie in the plane $x - 2y + z = 0$.

Ans. to Q. No. 11: The given lines are,

$$\frac{x+1}{3} = \frac{y-1}{3} = \frac{z-2}{-2} = k_1 \text{ (say)} \quad \text{... ... ...(i)}$$

and $$\frac{x-3}{1} = \frac{y-6}{2} = \frac{z+3}{-3} = k_2 \text{ (say)} \quad \text{... ... ...(ii)}$$

From (i) $x = 3k_1 - 1$, $y = 3k_1 + 1$, $z = 2 - 2k_1$;

thus $(3k_1 - 1, 3k_1 + 1, 2 - 2k_2)$ is an arbitrary point on line (i).

From (ii) $x = k_2 + 3$, $y = 2k_2 + 6$, $z = -3k_2 - 3$; thus,

$(k_2 + 3, 2k_2 + 6, -3k_2 - 3)$ is an arbitrary point on line (ii).

In order that the lines (i) and (ii) may intersect,these two points should be identical for some values of k_2 and k_2:

$3k_1 - 1 = k_2 + 3$ ... ...(iii)

$3k_1 + 1 = 2k_2 + 6$ ... ...(iv)

$2 - 2k_1 = -3k_2 - 3$ ... ...(v)

Solving (iii) and (iv), we get $k_2 = -1$ and $k_1 = 1$; also these values of k_1 and k_2 satisfy (v). Hence the two given lines (i) and (ii) intersect at the point whose coordinates are (3.1 – 1, 3.1 + 1, 2 – 2.1) i.e., (2, 4, 0). As the two lines intersect,they are coplanar.

Hence the given lines (i) and (ii) are coplanar.Their point of intersection is (2, 4, 0).

Ans. to Q. No. 12: $3x - 4y + 5 = 0 = 6x - 2y + z + 9 = 0$ ... ...(i)

Line through the origin parallel to the above line is given by

$3x - 4y = 0, 6x - 2y + z = 0$

$\Rightarrow 3x = 4y, 2.4y - 2y + z = 0$

$\Rightarrow 3x = 4y, 6y = -z$

$\Rightarrow \frac{x}{4} = \frac{y}{3}, \frac{y}{3} = -\frac{z}{18}$

$\Rightarrow \frac{x}{4} = \frac{y}{3} = -\frac{z}{18}$

This line has direction ratios 4, 3, –18 and so the line (i), which is its parallel, has direction ratios 4, 3, –18.

Putting y = 0 in (i), we get,

$3x + 5 = 0, 6x + z + 9 = 0$

$\Rightarrow x = -\frac{5}{3}, z = -6x - 9 = 10 - 9 = 1.$

Thus line (i) passes through the point $\left(-\frac{5}{3}, 0, 1\right)$.

The equations of the line (i) in symmetical form as,

$$\frac{x-\left(-\frac{5}{3}\right)}{4} = \frac{y-0}{3} = \frac{z-1}{-18} \quad \text{... ... ...(ii)}$$

Also z-axis is a line having direction cosines 0, 0, 1 passing through (0, 0, 0); so the equation of the z-axis in the symmetrical form are,

$$\frac{x-0}{0}=\frac{y-0}{0}=\frac{z-0}{1} \qquad \text{... ... ...(iii)}$$

The required S.D between the z-axis and the given line =

S.D between the lines (ii) and (iii) = $\frac{|\Delta|}{k}$

Where, $\Delta = \begin{vmatrix} 0-\left(-\frac{5}{3}\right) & 0-0 & 0-1 \\ 4 & 3 & -18 \\ 0 & 0 & 1 \end{vmatrix}$

$$= \begin{vmatrix} \frac{5}{3} & 0 & -1 \\ 4 & 3 & -18 \\ 0 & 0 & 1 \end{vmatrix}$$

$= 5$

Then we find k and find S.D (Do yourself) k = 5, S.D = 1.

Ans. to Q. No. 13: The given lines are, $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$ and

$$\frac{x-4}{2}=\frac{y-6}{3}=\frac{z-8}{4}$$

Here, $\Delta = \begin{vmatrix} 4-1 & 6-1 & 8-1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix}$

$= 3(8 – 9) – 5(4 – 6) + 7(3 – 4)$

$= –3 + 10 – 7 = 0$

and $k = \sqrt{\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}^2 + \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}^2 + \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}^2}$

$= \sqrt{1+1+4}$

$= \sqrt{6}$

Required S.D = $\frac{|\Delta|}{k} = \frac{0}{\sqrt{6}} . = 0$

S.D = 0 implies that the two lines are coplanar and intersecting.

Ans. to Q. No. 14: The line $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-5}{4}$ passes through the point $(\alpha, \beta, \gamma) = (2, 4, 5)$ and has direction ratios 1, 3, 4;

as $\sqrt{1^2+3^2+4^2} = \sqrt{26}$, its direction cosines l, m, n are given by

$l = \frac{1}{\sqrt{26}}, \; m = \frac{3}{\sqrt{26}}, \; n = \frac{4}{\sqrt{26}}$

If d be the distance of the given point $(x', y', z') = (1, 2, 3)$ from the above line, we have,

$$d^2 = \begin{vmatrix} x'-\alpha & y'-\beta \\ l & m \end{vmatrix}^2 + \begin{vmatrix} y'-\beta & z'-\gamma \\ m & n \end{vmatrix}^2 + \begin{vmatrix} z'-\gamma & x'-\alpha \\ n & l \end{vmatrix}^2$$

i.e., $d^2 = \begin{vmatrix} 1-2 & 2-4 \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} \end{vmatrix}^2 + \begin{vmatrix} 2-4 & 3-5 \\ \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \end{vmatrix}^2 + \begin{vmatrix} 3-5 & 1-2 \\ \frac{4}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{vmatrix}^2$

$$= \left(-\frac{3}{\sqrt{26}} + \frac{2}{\sqrt{26}}\right)^2 + \left(-\frac{8}{\sqrt{26}} + \frac{6}{\sqrt{26}}\right)^2 + \left(-\frac{2}{\sqrt{26}} + \frac{4}{\sqrt{26}}\right)^2$$

$$= \left(-\frac{1}{\sqrt{26}}\right)^2 + \left(-\frac{2}{\sqrt{26}}\right)^2 + \left(-\frac{2}{\sqrt{26}}\right)^2$$

$$= \frac{1}{26} + \frac{4}{26} + \frac{4}{26}$$

$$= \frac{9}{26}$$

$\therefore d = \frac{3}{\sqrt{26}}$

Ans. to Q. No. 15: The 1st line is given by $x = az + b$, $y = cz + d$

i.e., $x - b = az$, $y - d = cz$

i.e., $\frac{x-b}{a} = \frac{y-d}{c} = \frac{z-0}{1}$

The 2nd line is given by $x = a'z + b'$, $y = c'z + d'$

i.e., $\frac{x-b'}{a'} = \frac{y-d'}{c'} = \frac{z-0}{1}$

The two lines will be coplanar if, $\begin{vmatrix} b-b' & d-d' & 0 \\ a & c & 1 \\ a' & c' & 1 \end{vmatrix} = 0$

$$\text{i.e., } \begin{vmatrix} b-b' & d-d' & 0 \\ a-a' & c-c' & 0 \\ a' & c' & 1 \end{vmatrix} = 0 \quad (\text{by } R_2 - R_3)$$

i.e., $(b - b')(c - c') - (a - a')(d - d') = 0$

i.e., $(b - b')(c - c') = (a - a')(d - d')$

Which is the required condition.



8.8 MODEL QUESTIONS

Q.1: Find k so that the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y+3}{1} = \frac{z-6}{-5}$ are coplanar.

Q.2: Find the angle between the lines $\frac{x-6}{2} = \frac{y-2}{1} = \frac{z+4}{-1}$ and $x + 5y - 2z = 6$, $6x - 4y + 5z = 2$.

Q.3: Find the length of the perpendicular from the point (–1, 3, 9) to the line $\frac{x-3}{2} = \frac{y-1}{-5} = \frac{z+2}{3}$.

Q.4: Prove that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar and find the equation of the plane containing them.

Q.5: Find the length of the shortest distance between the lines

$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ and $\frac{x}{-3} = \frac{y+9}{2} = \frac{z-2}{4}$.

*** ***** ***

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1) Approximately how many hours did you spend for studying the units in the course? ☐☐☐

2) Please give your opinions (by ✓ mark) to the following items based on your reading of the block:

Sl. No.	Statements	Strongly Agree	Agree	Neither Agree nor Disagree	Disagree	Strongly Disagree
I)	The SLMs of the course fulfil the learning objectives set out in the beginning of each unit					
II)	The units of the SLM could satisfy my academic needs and expectations					
III)	The Unit writers have excellent knowledge about the course contents					
IV)	Language and contents of the units were presented at a level which I could readily understand					
V)	Ample opportunity for participation in the activities provided in the units					
VI)	Used enough Illustrations (Diagrams, tables etc.) for conceptual clarity					
VII)	Quality of content is engaging, relevant, and up-to-date					
VIII)	The self check questions are very helpful					
IX)	The Possible/model questions and the answers to check my progress have benefited me a lot					

Additional Comments: *(Please feel free to provide your open comments)*

1) Which aspects of the SLM, according to you, worked well?

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2) What sort of changes/improvements do you feel KKHSOU could implement to improve the overall quality of the SLM?

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Thank you for taking the time to complete this form.