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Coordinate Geometry and Vector Analysis

SEMESTER - V

MATHEMATICS

Block 2

KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

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CONTENTS

UNIT 9: Sphere **Pages: 205-217**

Equation to a Sphere with given Centre and given Radius, General Equation of Sphere, Diameter form of the Equation of a Sphere

UNIT 10: Cone **Pages: 218-233**

Equation of a Cone with a given Vertex and a given Guiding Curve, Locus of Second Degree Homogeneous Equation in x, y, z, Right Circular Cone

UNIT 11: Cylinder **Pages: 234-244**

Definition, Equation of Cylinder

UNIT 12: Introduction to Vectors **Pages: 245-267**

Vectors and Scalars, Types of Vectors, Operation on Vectors: Additions of Vectors, Subtraction of Vectors, Multiplication of Vectors by Scalars, Section Formula, Components of a Vector

UNIT 13: Product of Vectors-I **Pages: 268-281**

Product of Two Vectors: Scalar or Dot Product, Vector (Cross) Product

UNIT 14: Product of Vectors-II **Pages: 282-291**

Product of Three Vectors, Scalar Triple Product, Vector Triple Product, Product of Four Vectors, Scalar Product of Four Vectors, Vector Product of Four Vectors

UNIT 15: Vector Calculus **Pages: 292-315**

Scalar and Vector Point Function, Vector Differential Operator (Del), Gradient, Divergence and Curl and their Identities, Lapalacian Operator

BLOCK INTRODUCTION

This is the second block of the course “Coordinate Geometry and Vector analysis”. The block consists of seven units.

Units 9,10 & 11, covers the basics concept of sphere,cone and cylinder.

In Unit 12, begins with introduction vectors.

In Units 13 & 14, deals with product of vectors.

In Unit 15, covers the basic concept of vector calculus.

While going through a unit, you will notice some along-side boxes, which have been included to help you know some of the difficult, unseen terms. Again, we have included some relevant concepts in ‘‘LET US KNOW’’ along with the text. And, at the end of each section, you will get “CHECK YOUR PROGRESS” questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with “ANSWERS TO CHECK YOUR PPROGRESS” given at the end of each unit.

UNIT 9: SPHERE

UNIT STRUCTURE

9.1 Learning Objectives

9.2 Introduction

9.3 Sphere

9.3.1 Equation to a Sphere with given Centre and given Radius

9.3.2 General Equation of Sphere

9.3.3 Diameter form of the Equation of a Sphere

9.4 Let Us Sum Up

9.5 Further Reading

9.6 Answers to Check Your Progress

9.7 Model Questions

9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know about sphere and its equation.

9.2 INTRODUCTION

In unit 6, 7 and 8, we have already discussed the coordinates system in three dimension,equation of plane and line. In this unit, we will discuss about sphere and its equation.

9.3 SPHERE

Definition: The locus of a point which moves in space in such a way that it is always at a constant distance from a fixed point, is called a sphere.The fixed point is called the centre of the sphere and the constant distance is called its radius.

9.3.1 Equation to a Sphere with given Centre and given Radius

Let C(p, q, r) be the centre and ‘a’ the radius of a sphere. Let P(x, y, z) be any point on the sphere. Then, CP = a.

$\Rightarrow$ $CP^2 = a^2$

$\Rightarrow$ $(x - p)^2 + (y - q)^2 + (z - r)^2 = a^2$

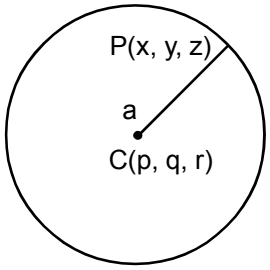


Fig. 9.1

As this equation is satisfied by the coordinate of any point P on the sphere, it is the equation of the sphere.

Note: The equation of the sphere whose centre is at the origin and radius is a, is $x^2 + y^2 + z^2 = a^2$.

9.3.2 General Equation of Sphere

Consider the equation: $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Where, u, v, w and d are arbitrary constants.

The equation can be expressed as:

$(x^2 + 2ux + u^2) + (y^2 + 2vy + v^2) + (z^2 + 2wz + w^2) = u^2 + v^2 + w^2 - d$

i.e., $(x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d$

i.e., $\{x - (-u)\}^2 + \{y - (-v)\}^2 + \{z - (-w)\}^2 = \left(\sqrt{u^2 + v^2 + w^2 - d}\right)^2$

and therefore represents a sphere whose centre in the point (–u, –v, –w) and radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

Note:

i) The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is called the general equation of a sphere, it contains four independent arbitrary constants u, v, w and d and so to know a particular sphere four independent conditions are necessary.

ii) The sphere represented by the above equation is real if $u^2 + v^2 + w^2 \geq d$ and imaginary if $u^2 + v^2 + w^2 > d$. However in the real situation it reduces to the point (–u, –v, –w) if $u^2 + v^2 + w^2 = d$.

iii) It is easy to see that an equation of the form:

$ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0$

Which is the same as:

$x^2 + y^2 + z^2 + 2\left(\frac{u}{a}\right)x + 2\left(\frac{v}{a}\right)y + 2\left(\frac{w}{a}\right)z + \frac{d}{a} = 0$

represents a sphere whose centre is $\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)$ and radius is $\sqrt{\frac{u^2}{a^2} + \frac{v^2}{a^2} + \frac{w^2}{a^2} - \frac{d}{a^2}}$.

9.3.3 Diameter form of the Equation of a Sphere

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the extremities of a diameter of a sphere. Let P(x, y, z) be any point on the sphere. Then $\angle APB$ is a right angle.

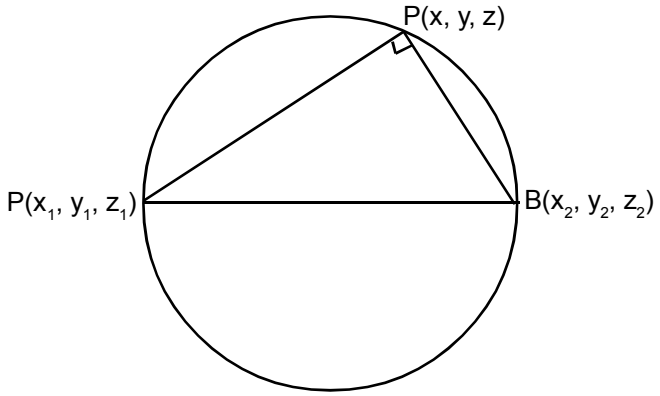


Fig. 9.2

Now, AP has direction ratios $x - x_1, y - y_1, z - z_1$ and BP has direction ratios: $x - x_2, y - y_2, z - z_2$.

As AP and BP are at right angles, we have,

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

Which is the required equation of the sphere.

Equation of a sphere passing through a given circle: It is easy to see that the section of a sphere by a plane is a circle. If S = 0 is the equation of a sphere and L = 0 is that of a plane, then the two equations S = 0 & L = 0 taken together represent a circle, viz, the circle in which the plane L = 0 cuts the sphere S = 0. Given a circle represented by the equations S = 0 and L = 0, then the equation $S + \lambda L = 0$ represents a sphere passing through the circle. For different

values of the arbitrary constant λ, we get different spheres passing through the circle.

ILLUSTRATIVE EXAMPLES:

Example 1: Find the equation of the sphere whose centre is (–1, 3, 2) and radius is 5.

Solution: The equation of the sphere,whose centre is (p, q, r) and radius is a, is: $(x - p)^2 + (y - q)^2 + (z - r)^2 = a^2$.

Hence the required equation of the sphere whose centre is (–1, 3, 2) and radius is 5, is:

$$\{x - (-1)\}^2 + (y - 3)^2 + (z - 2)^2 = 5^2$$

i.e., $(x + 1)^2 + (y - 3)^2 + (z - 2)^2 = 25$

i.e., $(x^2 + 2x + 1) + (y^2 - 6y + 9) + (z^2 - 4z + 4) = 25$

i.e., $x^2 + y^2 + z^2 + 2x - 6y - 4z - 11 = 0$

Example 2: Find the centre and radius of the sphere given by $2(x^2 + y^2 + z^2) - 2x + 4y - 6z - 15 = 0$.

Solution: The equation of the sphere is:

$$2(x^2 + y^2 + z^2) - 2x + 4y - 6z - 15 = 0$$

i.e., $x^2 + y^2 + z^2 - x + 2y - 3y - \frac{15}{2} = 0$

Comparing with the general equation of a sphere viz., $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we have,

$$2u = -1,\ 2v = 2,\ 2w = -3,\ d = -\frac{15}{2}$$

$$\Rightarrow u = -\frac{1}{2},\ v = 1,\ w = -\frac{3}{2},\ d = -\frac{15}{2}$$

The centre of the sphere has coordinates (–u, –v, –w)

i.e., $\left(-\frac{1}{2}, 1, -\frac{3}{2}\right)$ and its radius $= \sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{\frac{1}{4} + 1 + \frac{9}{4} + \frac{15}{2}}$$

$$= \sqrt{\frac{1 + 4 + 9 + 30}{4}}$$

$$= \sqrt{11}$$

Example 3: Show that the point P(–2, 5, 7) doesnot lie on the sphere $x^2 + y^2 + z^2 – 2x – 4y – 6z + 10 = 0$. Determine whether P lies inside or outside the sphere.

Solution: When x = –2, y = 5 , z = 7, we have,

$x^2 + y^2 + z^2 – 2x – 4y – 6z + 10$

$= 4 + 25 + 49 + 4 – 20 – 42 + 10$

$= 92 – 62 = 30 \neq 0$

So the equation is not satisfied and hence the point P(–2, 5, 7) doesnot lie on the sphere.Now the centre of the sphere:

$x^2 + y^2 + z^2 – 2x – 4y – 6z + 10 = 0$ is the point C(1, 2, 3) and its radius is: $r = \sqrt{1^2 + 2^2 + 3^2 – 10} = \sqrt{14 – 10} = \sqrt{4} = 2$

We have, $CP = \sqrt{(1+2)^2 + (2-5)^2 + (3-7)^2}$

$= \sqrt{9 + 9 + 16}$

$= \sqrt{34} > 2$

i.e., CP > r; it follows that P lies outside the sphere.

Example 4: Find the equation of the sphere standing on the join of (2, –3, 1) and (3, –1, 2) as diameter.

Show that the point $P\left(\frac{5}{2}, -2, \frac{1}{2}\right)$ lies inside the sphere.

Solution: The sphere stands on the join of (2, –3, 1) and (3, –1, 2) as diameter. Its equation is:

$(x – 2)(x – 3) + \{y – (–3)\}\{y – (–1)\} + (z – 1)(z – 2) = 0$

i.e., $(x – 2)(x – 3) + (y + 3)(y + 1) + (z – 1)(z – 2) = 0$

i.e., $(x^2 – 5x + 6) + (y^2 + 4y + 3) + (z^2 – 3z + 2) = 0$

i.e., $x^2 + y^2 + z^2 – 5x + 4y – 3z + 11 = 0$

Its centre is the point $C\left(\frac{5}{2}, -2, \frac{3}{2}\right)$ and radius is:

$$\sqrt{\left(-\frac{5}{2}\right)^2 + 2^2 + \left(-\frac{3}{2}\right)^2 – 11}$$

i.e., $\sqrt{\frac{17}{2} – 7}$ i.e., $\sqrt{\frac{3}{2}}$.

Consider the point $P\left(\frac{5}{2}, -2, \frac{1}{2}\right)$, we have,

$CP^2 = \left(\frac{5}{2} - \frac{5}{2}\right)^2 + (-2+2)^2 + \left(\frac{3}{2} - \frac{1}{2}\right)^2$

$= 1 < \frac{3}{2}$

So that, CP > $\sqrt{\frac{3}{2}}$

i.e., CP < radius of the sphere.

$\therefore$ $P\left(\frac{5}{2}, -2, \frac{1}{2}\right)$ lies inside the sphere.

Example 5: Find the equation of the sphere cutting off intercepts a, b, c on the coordinate axes and passing through the origin. Find its centre and radius.

Solution: According to the question,we have to find the equation of the sphere passing through the four points (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c).

Let the equation of the sphere be:

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... ...(i)

As the sphere passes through the points (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c), their coordinates must satisfy the equation (i).

Then we get,

$d = 0$, $a^2 + 2ua + d = 0$, $b^2 + 2vb + d = 0$, $c^2 + 2wc + d = 0$

$\Rightarrow d = 0$, $2ua = -a^2$, $2vb = -b^2$, $2wc = -c^2$ (using d = 0)

$\Rightarrow u = -\frac{a}{2}$, $v = -\frac{b}{2}$, $w = -\frac{c}{2}$, $d = 0$

Hence (i) becomes: $x^2 + y^2 + z^2 - ax - by - cz = 0$

Which is the required equation of the sphere.

The centre of the sphere is the point (–u, –v, –w) i.e., $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ and

its radius = $\sqrt{u^2 + v^2 + w^2 - d}$

$= \frac{1}{2}\sqrt{a^2 + b^2 + c^2}$.

Example 6: Find the equation of the sphere passing through the circle $x^2 + y^2 + z^2 - 4x + 6y - 11 = 0$, $2x - 3y + 5z + 4 = 0$ and the point (1, 1, 1).

Solution: The equation of any sphere passing through the circle $x^2 + y^2 + z^2 - 4x + 6y - 11 = 0$, $2x - 3y + 5z + 4 = 0$ is:

$x^2 + y^2 + z^2 - 4x + 6y - 11 + \lambda(2x - 3y + 5z + 4) = 0$

If it passes through the point (1, 1, 1), then we have,

$1 + 1 + 1 - 4 + 6 - 11 + \lambda(2 - 3 + 5 + 4) = 0$

$\Rightarrow 8\lambda = 6$

$\Rightarrow \lambda = \frac{6}{8} = \frac{3}{4}$

Therefore the required equation of the desired sphere is:

$$x^2 + y^2 + z^2 - 4x + 6y - 11 + \frac{3}{4}(2x - 3y + 5z + 4) = 0$$

i.e., $4x^2 + 4y^2 + 4z^2 - 16x + 24y - 44 + 6x - 9y + 15z + 12 = 0$

i.e., $4x^2 + 4y^2 + 4z^2 - 10x + 15y + 15z - 32 = 0$



CHECK YOUR PROGRESS

Q.1: Find the equation of the sphere whose centre is (2, 0, –5) and radius is 11.

Q.2: Find the centre and radius of the sphere:

i) $x^2 + y^2 + z^2 - 16x - 6y = 0$

ii) $3(x^2 + y^2 + z^2) + x - 5y - 2z + 1 = 0$

Q.3: Determine whether the point P(1,2,-1) lies inside,on or outside of the sphere $x^2 + y^2 + z^2 - 4x + 6y - 3 = 0$.

Q.4: Find the equation of the sphere described on the join of (2, –3, 4) and (–2, 1, –5) as a diameter.

Q.5: The plane $x - 2y + 3z = 12$ cuts the axes in A, B, C. Find the equation of the sphere OABC.

Q.6: Find the equation of the sphere passing the points (1, 0, 0), (0, 1, –1), (–1, 2, 0) and (1, 2, 3).

Q.7: Find the equation of the sphere on which the circle given by $2x + y + z + 3 = 0$ and $x^2 + y^2 + z^2 = 9$ is a great circle.
[The circle in which a plane passing through the centre of a sphere intersects the sphere is called a great circle of the sphere.]



9.4 LET US SUM UP

- The equation of the sphere whose centre is at the origin and radius is a, is: $x^2 + y^2 + z^2 = a^2$.
- The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere whose centre is the point (–u, –v, –w) and radius is $\sqrt{u^2 + v^2 + w^2 - d}$.
- The equation of the sphere on the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$.



9.5 FURTHER READING

1) Jain, P. K. & Ahmad, Khalil. *Textbook of Analytical Geometry of Three Dimensions*. New Delhi: New Age International (P) Limited.
2) Khan, R. M. *Analytical Geometry of Two & Three Dimensions & Vector Analysis*. New Central Book Agency (P) Limited.
3) Kishan, Hari. *Coordinate Geometry of Two Dimensions*. New Delhi: Atlantic Publishers & Distributors.
4) Pal, Sat & Pal, Harbans. *A Textbook of Two Dimensional Geometry*. New Age International Publishers.
5) Sharma, G. C. & Jain, Madhu. *Coordinate Geometry (2D and 3D)*. Galgotia Publications Pvt Lid.
6) Sharma, A. K. *Coordinate Geometry (2D)*. New Delhi: Discovery Publishing House.
7) Vasistha, A. R. & Agrawal, *D. C. Analytical Geometry of Three Dimensions (Coordinate of Solid Geometry)*.

9.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: The equation of the sphere is:

$$(x - 2)^2 + (y - 0)^2 + (z + 5)^2 = 11^2$$

i.e., $x^2 - 4x + 4 + y^2 + z^2 + 10z + 25 = 121$

i.e., $x^2 + y^2 + z^2 - 4x + 10z - 92 = 0$

Ans. to Q. No. 2: i) Equation of the sphere is: $x^2 + y^2 + z^2 - 16x - 6y = 0$

Comparing with the general equation of a sphere viz.,

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we have,

$2u = -16, 2v = -6, 2w = 0, d = 0$

$\Rightarrow u = -8, v = -3, w = 0, d = 0$

$\therefore$ The centre of the sphere is the point $(-u, -v, -w)$

i.e., $(8, 3, 0)$ and its radius $= \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{73}$

ii) The equation of the sphere is:

$$3(x^2 + y^2 + z^2) + x - 5y - 2z + 1 = 0$$

i.e., $x^2 + y^2 + z^2 + \frac{1}{3}x - \frac{5}{3}y - \frac{2}{3}z + \frac{1}{3} = 0$

Here, in usual notations,

$$2u = \frac{1}{3}, 2v = -\frac{5}{3}, 2w = -\frac{2}{3}, d = \frac{1}{3}$$

$$\Rightarrow u = \frac{1}{6}, v = -\frac{5}{6}, w = -\frac{1}{3}, d = \frac{1}{3}$$

$\therefore$ Coordinates of the centre are $(-u, -v, -w)$ i.e., $\left(-\frac{1}{6}, \frac{5}{6}, \frac{1}{3}\right)$.

Also radius $= \sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{\left(\frac{1}{6}\right)^2 + \left(-\frac{5}{6}\right)^2 + \left(-\frac{1}{3}\right)^2 - \frac{1}{3}}$$

$$= \sqrt{\frac{1 + 25 + 4 - 12}{36}}$$

$$= \sqrt{\frac{18}{36}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Ans. to Q. No. 3: For the sphere $x^2 + y^2 + z^2 - 4x + 6y - 3 = 0$

We have, $2u = -4$, $2v = 6$, $2w = 0$, $d = -3$

$\Rightarrow u = -2$, $v = 3$, $w = 0$, $d = -3$

$\therefore$ Its centre C(say) has coordinates $(-u, -v, -w)$

i.e., $(2, -3, 0)$ and its radius is: $r = \sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{4 + 9 + 0 + 3}$

$= \sqrt{16}$

$= 4$

The given point is $P(1, -2, 1)$

Now $CP = \sqrt{(1-2)^2 + (-2+3)^2 + (1-0)^2}$

$= \sqrt{1+1+1}$

$= \sqrt{3} < 4$

i.e., $CP < r$.

$\therefore$ P(1, –2, 1) lies inside the sphere.

Ans. to Q. No. 4: The required equation of the sphere is:

$(x - 2)(x + 2) + (y + 3)(y - 1) + (z - 4)(z + 5) = 0$

i.e., $(x^2 - 4) + (y^2 + 2y - 3) + (z^2 + z - 20) = 0$

i.e., $x^2 + y^2 + z^2 + 2y + z - 327 = 0$

Ans. to Q. No. 5: The plane is $x - 2y + 3z = 12$

i.e., $\frac{x}{12} - \frac{2y}{12} + \frac{3z}{12} = \frac{12}{12}$

i.e., $\frac{x}{12} + \frac{y}{-6} + \frac{z}{4} = 1$

This plane cuts the axes at A, B, C, so A, B, C have coordinates (12, 0, 0), (0, –6, 0), (0, 0, 4) respectively.

Let the equation of the sphere OABC be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... ...(1)

As it passes through the points O(0, 0, 0), A(12, 0, 0), B(0, –6, 0) and C(0, 0, 4), their coordinates must satisfy (1). So we get,

$d = 0$, $144 + 24u + d = 0$, $36 - 12v + d = 0$, $16 + 8w + d = 0$

i.e., $d = 0$, $144 + 24u = 0$, $36 - 12v = 0$, $16 + 8w = 0$

i.e., $d = 0$, $24u = -144$, $-12v = -36$, $8w = -16$

i.e., $d = 0$, $u = -6$, $v = 3$, $w = -2$

Putting the values u, v, w and d in (1), we get,

$x^2 + y^2 + z^2 - 12x + 6y - 4z = 0$, which is the required equation of the sphere.

Ans. to Q. No. 6: Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... ...(1)

As this sphere passes through the points

(1, 0, 0), (0, 1, –1), (–1, 2, 0) and (1, 2, 3), we get,

$$\left.\begin{aligned} &1 + 2u + d = 0 \\ &1 + 1 + 2v - 2w + d = 0 \\ &1 + 4 - 2u + 4v + d = 0 \\ &1 + 4 + 9 + 2u + 4v + 6w + d = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} &1 + 2u + d = 0 && \text{... ... ...(2)} \\ &2 + 2v - 2w + d = 0 && \text{... ... ...(3)} \\ \text{i.e.,}\ &5 - 2u + 4v + d = 0 && \text{... ... ...(4)} \\ &14 + 2u + 4v + 6w + d = 0 && \text{... ... ...(5)} \end{aligned}\right\}$$

(3) – (2), (4) – (2) , (5) – (2) give us respectively

$$\left.\begin{aligned} &-2u + 2v - 2w + 1 = 0 \\ &-4u + 4v + 4 = 0 \\ &4v + 6w + 13 = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} &2u - 2v + 2w = 1 \\ \text{i.e.,}\ &u - v + 0.w = 1 \\ &0.u + 4v + 6w = -13 \end{aligned}\right\}$$

From the above system, we get by Cramer's rule,

$$\frac{u}{\begin{vmatrix} 1 & -2 & 2 \\ 1 & -1 & 0 \\ -13 & 4 & 6 \end{vmatrix}} = \frac{v}{\begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & -13 & 6 \end{vmatrix}} = \frac{w}{\begin{vmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 4 & -13 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & -2 & 2 \\ 1 & -1 & 0 \\ 0 & 4 & 6 \end{vmatrix}}$$

$$\Rightarrow \frac{u}{1(-6-0)-1(-12-8)-13(0+2)} = \frac{v}{2(6-0)-1(6+26)}$$

$$= \frac{w}{2(13-4)-1(26-4)}$$

$$= \frac{1}{2(-6-0)-1(-12-8)}$$

$$\Rightarrow \frac{u}{-6+20-26} = \frac{v}{12-32}$$

$$= \frac{w}{18-22}$$

$$= \frac{1}{-12+20}$$

$$\Rightarrow \frac{u}{-12} = \frac{v}{-20} = \frac{w}{-4} = \frac{1}{8}$$

$$\Rightarrow u = -\frac{3}{2}, v = -\frac{5}{2}, w = -\frac{1}{2}$$

Hence (2) $\Rightarrow 1 + 2 \times \left(-\frac{3}{2}\right) + d = 0$

$\Rightarrow 1 - 3 + d = 0$

$\Rightarrow d = 2$

On putting the values of u, v, w and d (1) becomes

$x^2 + y^2 + z^2 - 3x - 5y - z + 2 = 0$

This is the required sphere.

Ans. to Q. No. 7: As the required sphere passes through the circle given by $2x + y + z + 3 = 0$ and $x^2 + y^2 + z^2 - 9 = 0$, its equation is of the form $(x^2 + y^2 + z^2 - 9) + k(2x + y + z + 3) = 0$.

i.e., $x^2 + y^2 + z^2 + 2kx + ky + kz + (3k - 9) = 0$ ... ...(1)

The centre of the sphere is $\left(-k, -\frac{k}{2}, -\frac{k}{2}\right)$.

As the given circle is a great circle of the above sphere,the plane of the circle viz.,

$2x + y + z + 3 = 0$ passes through the centre of the sphere;

So we get,

$$2(-k) + \left(-\frac{k}{2}\right) + \left(-\frac{k}{2}\right) + 3 = 0$$

$\Rightarrow -3k + 3 = 0$

$\Rightarrow -3k = -3$

$\Rightarrow k = 1$

Putting $k = 1$ in (1), we get, $x^2 + y^2 + z^2 + 2x + y + z - 6 = 0$

This is the required equation of the sphere.



9.7 MODEL QUESTIONS

Q.1: Find the equation of the sphere whose centre is (2, –3, 1) and radius 6.

Q.2: Find the equation of the sphere with centre (2, –2, 3) and passing through (7, –3, 5).

Q.3: Find the centre and radius of the sphere

i) $x^2 + y^2 + z^2 + 2x – 4y – 6z + 5 = 0$

ii) $2x^2 + 2y^2 + 2z^2 – 2x + 4y + 2z + 3 = 0$

Q.4: Find the equation of the sphere through the four points (1, 2, 3), (0, –2, 4), (4, –4, 2) and (3, 1, 4).

Q.5: Obtain the equation of the sphere passing through the points (3, 0, 2). (–1, 1, 1), (2, –5, 4) and having its centre on the plane $2x + 4y + 2z = 6$.

Q.6: Find the equation of the sphere through the circle through the points (4, –1, 2), (0, –2, 3), (1, 5, –1), (2, 0, 1).

Q.7: Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and passing through the origin.

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UNIT 10: CONE

UNIT STRUCTURE

10.1 Learning Objectives

10.2 Introduction

10.3 Cone

10.3.1 Equation of a Cone with a given Vertex and a given Guiding Curve

10.3.2 Locus of Second Degree Homogeneous Equation in x, y, z

10.3.3 Right Circular Cone

10.4 Let Us Sum Up

10.5 Answers to check your progress

10.6 Further Reading

10.7 Model Questions

10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know about cone and its equation.

10.2 INTRODUCTION

In unit 9, we discussed the sphere and its equation. In this unit, we will discuss about cone and its equation.

10.3 CONE

Definition: A cone is a surface generated by a straight line which always passes through a fixed point and satisfies another condition, say intersects a given curve. The moving line is called a generator, the fixed point is called the vertex and the given curve is called the guiding (or base) curve of the cone.

10.3.1 Equation of a Cone with a given Vertex and a given Guiding Curve

To find the equation of the cone whose vertex is (α, β, γ) and base is the curve z = 0, f(x, y) = 0.

The equations to any line through the point (α, β, γ) are:

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{... ... ...(1)}$$

This line meets the plane z = 0, when,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = -\frac{\gamma}{n}$$

$\Rightarrow x = \alpha - \frac{l\gamma}{n}$, $y = \beta - \frac{m\gamma}{n}$

Thus the line (1) meets the plane z = 0 at the point

$$\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$$

If the line (1) meets the curve z = 0, f(x, y) = 0 ... ...(2)

We must have $f\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}\right) = 0$ ... ...(3)

If we eliminate l, m, n between (1) and (3) , we obtain the equation to the locus of the variable line (1) always passing through (α, β, γ) and intersecting the curve (2), i.e., the equation of the cone whose vertex is the point (α, β, γ) and base is the curve (2).

To eliminate l, m, n from (1) and (3), we see that from (1)

$\frac{l}{n} = \frac{x-\alpha}{z-\gamma}$ and $\frac{m}{n} = \frac{y-\beta}{z-\gamma}$;

Substituting these values of $\frac{l}{n}$ and $\frac{m}{n}$ in (3), we get,

$$f\left(\alpha - \frac{\gamma(x-\alpha)}{z-\gamma}, \beta - \frac{\gamma(y-\beta)}{z-\gamma}\right) = 0$$

i.e., $f\left(\frac{\alpha z - \gamma x}{z-\gamma}, \frac{\beta z - \gamma y}{z-\gamma}\right) = 0$

Which is the required equation of the cone.

10.4.2 Locus of Second Degree Homogeneous Equation in x, y, z

The most general homogeneous equation of 2nd degree in x, y and z is:

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ ... ...(1)

Clearly origin 0 is a point on the surface represented by this equation.

If $P(x', y', z')$ be any point on the surface represented by (1), then we have, $ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' = 0$

Whence, $k^2(ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y') = 0$

i.e., $a(kx')^2 + b(ky')^2 + c(kz')^2 + 2f(ky')(kz') + 2g(kz')(kx') + 2h(kx')(ky') = 0$

for all values of k; this shows that the point (kx', ky', kz') is also a point on the surface (1) for all values of k. But (kx', ky', kz') is an arbitrary point on the variable line OP (whose equations are: $\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'}$) .

So if P is any point on the surface (1), every point of the line OP is also a point on the surface (1) i.e., the entire line OP lies on the surface (1). In other words (1) represents a cone generated by the variable line OP which always passes through the fixed point O so that O is the vertex of the cone. Thus the locus of the equation (1) is a cone whose vertex is at the origin.

10.4.3 Right Circular Cone

If the generator of a cone in all its positions, makes a constant angle with a fixed line through the vertex of the cone, then the fixed line is called the axis and the constant angle is called the semi-vertical angle of the cone. The cone is known as the Right circular cone.

Let $A(\alpha, \beta, \gamma)$ be the vertex, the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ be the axis and θ be the semi-vertical angle of a right circular cone; to find its equation.

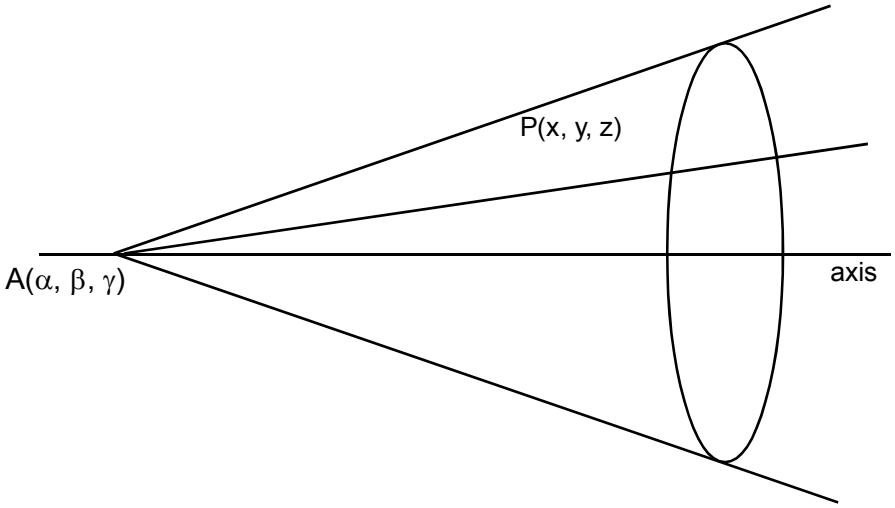


Fig. 10.1

Let P(x, y, z) be any point on the right circular cone. The generator AP makes constant angle θ with the axis of the cone; as the line AP has direction ratios $x - \alpha$, $y - \beta$, $z - \gamma$ and the axis has direction ratios l, m, n, we have,

$$\cos\theta = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

So that,

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = (l^2+m^2+n^2)\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}\cos^2\theta$$

This is true for the coordinates (x, y, z) of any point on the cone and hence this is the equation of the right circular cone.

Corollary: The equation of the right circular cone whose semi-vertical angle is θ, vertex at the origin and axis has direction ratios l, m, n is:

$(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2) = \theta$

This follows at once from the main result because $(\alpha, \beta, \gamma) = 0, 0, 0$.

ILLUSTRATIVE EXAMPLES:

Example 1: Find the equation of the cone whose vertex is (0, 0, 3) and base is the curve $x^2 - y^2 = 5$, $z = 0$.

Solution: Any line through the vertex (0, 0, 3) is:

$$\frac{x}{l} = \frac{y}{m} = \frac{z-3}{n} \qquad \text{... ... ...(1)}$$

and it intersects the plane $z = 0$ at the point where $z = 0$ and

$$\frac{x}{l} = \frac{y}{m} = \frac{-3}{n}$$

$$\Rightarrow x = -\frac{3l}{n},\ y = -\frac{3m}{n}$$

Thus the line (1) intersects the plane $z = 0$ at the point $\left(-\frac{3l}{n}, -\frac{3m}{n}, 0\right)$.

The line (1) will be a generator of the required cone if the above point lies on the base curve $z = 0$, $x^2 - y^2 = 5$ i.e., if

$$\left(-\frac{3l}{n}\right)^2 - \left(-\frac{3m}{n}\right)^2 = 5$$

i.e., if $9\left[\left(\frac{l}{n}\right)^2 - \left(\frac{m}{n}\right)^2\right] = 5$ ... ...(2)

Eliminating l, m and n from (1) and (2), we shall get the equation of the cone whose vertex is (0, 0, 3) and base is the curve $x^2 - y^2 = 5$, $z = 0$.

Now from (1) $\frac{l}{n} = \frac{x}{z-3}$, $\frac{m}{n} = \frac{y}{z-3}$;

Substituting these in (2), we get,

$$9\left[\left(\frac{x}{z-3}\right)^2 - \left(\frac{y}{z-3}\right)^2\right] = 5$$

i.e., $9(x^2 - y^2) = 5(z - 3)^2$ as the required equation of the cone.

Example 2: Find the equation of the cone with vertex at the origin which passes through the curve given by

$ax^2 + by^2 = 2z$, $lx + my + nz = p$.

Solution: The curve is given by $ax^2 + by^2 = 2z$ ... ...(1)

$lx + my + nz = p$ ... ...(2)

(2) can be written as $\frac{lx + my + nz}{p} = 1$ and with its help we make (1)

homogeneous: $ax^2 + by^2 - 2z\left(\frac{lx + my + nz}{p}\right)$ ... ...(3)

Now the equation (3) is homogeneous and so it represents a cone with vertex at the origin. Also (3) is satisfied if both (1) and (2) are

satisfied; this means that every point on the curve given by (1) and (2) lies on the surface (3), i.e., the cone given by (3), whose vertex is at the origin, passes through the given curve .

Hence the required equation of the cone is:

$p(ax^2 + by^2) = 2z(lx + my + nz)$.

Example 3: Show that the equation of the cone whose vertex is the origin and base the curve z = k, f(x, y) = 0 is $f\left(\frac{xk}{z}, \frac{yk}{z}\right) = 0$.

Solution: Any line through the vertex (0, 0, 0) of the cone is given by

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad \text{... ... ...(1)}$$

The line intersects the plane z = k at the point where z = k and

$$\frac{x}{l} = \frac{y}{m} = \frac{k}{n}$$

$$\Rightarrow x = \frac{kl}{n}, y = \frac{km}{n}$$

Thus the line (1) intersects the plane z = k at the point $\left(\frac{kl}{n}, \frac{km}{n}, k\right)$.

The line (1) will be a generator of the cone if it intersects the base curve z = k, f(x, y) = 0 for which the point $\left(\frac{kl}{n}, \frac{km}{n}, k\right)$ must be on that curve i.e., if $f\left(\frac{kl}{n}, \frac{km}{n}\right) = 0$ ... ...(2)

The required equation of the cone will be obtained by eliminating l, m and n from (1) and (2), which is done below.

From (1) $\frac{l}{n} = \frac{x}{z}$ and $\frac{m}{n} = \frac{y}{z}$;

Putting these expressions for $\frac{l}{n}$ and $\frac{m}{n}$ in (1), we get,

$$f\left(\frac{xk}{z}, \frac{yk}{z}\right) = 0$$

Which is the required equation of the cone.

Example 4: Find the equation of the right circular cone whose vertex is the origin, axis is the z-axis and semi-vertical angle is α.

Solution: The origin O is the vertex and OX is the axis of the right circular cone. Let P(x, y, z) be any point on the right circular cone.

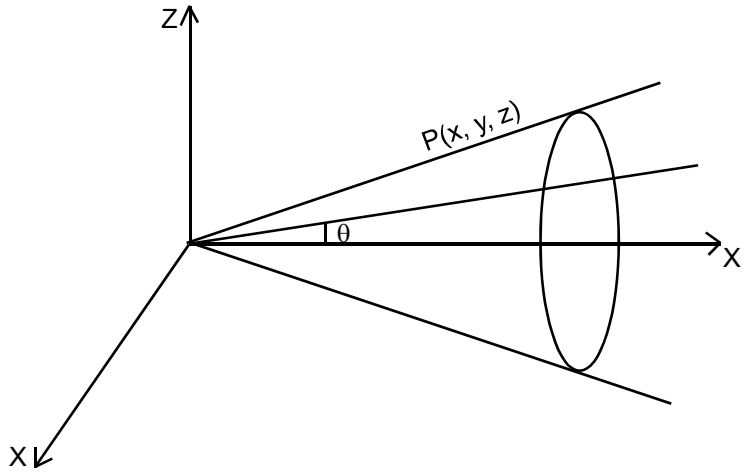


Fig. 10.2

Then OP is a generator having direction ratios x y, z and OX has direction cosines 1, 0, 0. As α is the semi-vertical angle of the right, circular cone, OP makes angle α with OX. So we get,

$$\cos\alpha = \frac{x.1 + y.0 + z.0}{\sqrt{x^2 + y^2 + z^2}\sqrt{1^2 + 0^2 + 0^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\Rightarrow \cos^2\alpha = \frac{x^2}{x^2 + y^2 + z^2}$$

$$\Rightarrow x^2 + y^2 + z^2 = \frac{x^2}{\cos^2\alpha} = x^2\sec^2\alpha$$

$$\Rightarrow y^2 + z^2 = x^2(\sec^2\alpha - 1)$$

$$\Rightarrow y^2 + z^2 = x^2\tan^2\alpha$$

Thus the required equation of the right circular cone is:

$y^2 + z^2 = x^2\tan^2\alpha$

Alternative Method: Let O be the vertex and OX be the axis of the right circular cone. Let P(x, y, z) be any point on the right circular cone and PN $\perp$ OX. Then ON = x and $\angle$XOP = α as α is the semi-vertical angle of the cone. Now from the right angled triangle OPN.

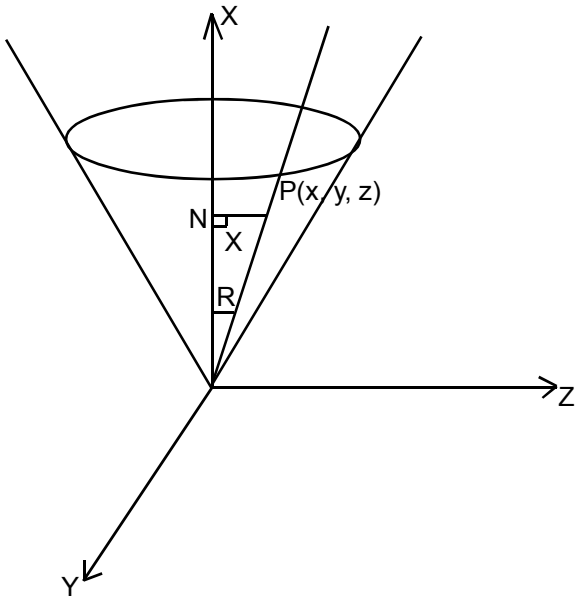


Fig. 10.3

We have, $\frac{OP}{ON} = \sec\alpha$

$\Rightarrow \frac{\sqrt{x^2 + y^2 + z^2}}{x} = \sec\alpha$

$\Rightarrow x^2 + y^2 + z^2 = x^2\sec^2\alpha$

$\Rightarrow y^2 + z^2 = x^2(\sec^2\alpha - 1)$

$\Rightarrow y^2 + z^2 = x^2\tan^2\alpha$

Thus the required equation of the right circular cone is:

$y^2 + z^2 = x^2\tan^2\alpha$.

Example 5: Find the equation of the right circular cone whose vertex is at (1, –2, –1), semi-vertical angle 60^0 and axis is the line,

$$\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z+1}{5}$$

Solution: We know that the right circular cone whose vertex is at (α, β, γ), semi-vertical angle is θ and axis is the line,

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ has equation,

$$[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2 = (l^2 + m^2 + n^2)\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}\cos^2\theta$$

For the given right circular cone vertex is at (1, –2, –1), semi-vertical angle 60^0 and axis is the line, $\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z+1}{5}$;

hence the equation of the cone is:

$[3(x-1) + (-4)(y+2) + 5(z+1)]^2$

$= [3^2 + (-4)^2 + 5^2][(x-1)^2 + (y+2)^2 + (z+1)^2]\cos^2 60^0$

i.e., $(3x - 4y + 5z - 6)^2 \times [(x-1)^2 + (y+2)^2 + (z+1)^2] \times \left(\frac{1}{2}\right)^2$

i.e., $2(3x - 3y + 5z - 6)^2 = 25[(x-1)^2 + (y+2)^2 + (z+1)^2]$



CHECK YOUR PROGRESS

Q.1: Find the equation of the cone whose vertex is (α, β, γ) and the guiding curve is the conic z = 0, $ax^2 + by^2 = 1$.

Q.2: Find the equation of the cone whose vertex is (α, β, γ) and base is the parabola y = 0, $x^2 = 4az$.

Q.3: Find the equation of the right circular cone whose vertex is the origin, axis is the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and semi-vertical angle is 45^0.

Q.4: Find the equation of the right circular cone whose vertex is (1, 0, –1), axis is the line $\frac{x-1}{2} = \frac{y}{-1} = \frac{z+1}{2}$ and semi-vertical angle is 30^0.

Q.5: Find the equation of the cone whose vertex is at the origin and which passes through the curve:

i) $ax^2 + by^2 = 1$, z = k

ii) 8yz + 7zx – 3y + 2 = 0, 3z – 2y – 5 = 0.

Q.6: Find the equation of the right circular cone:

i) whose vertex is at the origin,axis is OY and semi-vertical angle is 45^0.

ii) whose vertex is at the origin, axis is OZ and semi-vertical angle is α.



10.4 LET US SUM UP

- The second degree homogeneous equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a cone whose vertex is the origin.
- The equation of the right circular cone whose vertex is at (α, β, γ), semi-vertical angle θ, and axis the line,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \text{ is:}$$

$$[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2 = (l^2 + m^2 + n^2)\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}\cos^2\theta$$

- The equation of the right circular cone whose semi-vertical angle is θ, vertex at the origin and axis has direction ratios l, m, n is:
  $(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2)\cos^2\theta$.



10.5 FURTHER READING

1) Jain, P. K. & Ahmad, Khalil. *Textbook of Analytical Geometry of Three Dimensions*. New Delhi: New Age International (P) Limited.
2) Khan, R. M. *Analytical Geometry of Two & Three Dimensions & Vector Analysis*. New Central Book Agency (P) Limited.
3) Kishan, Hari. *Coordinate Geometry of Two Dimensions*. New Delhi: Atlantic Publishers & Distributors.
4) Pal, Sat & Pal, Harbans. *A Textbook of Two Dimensional Geometry*. New Age International Publishers.
5) Sharma, G. C. & Jain, Madhu. *Coordinate Geometry (2D and 3D)*. Galgotia Publications Pvt Lid.
6) Sharma, A. K. *Coordinate Geometry (2D)*. New Delhi: Discovery Publishing House.
7) Vasistha, A. R. & Agrawal, *D. C. Analytical Geometry of Three Dimensions (Coordinate of Solid Geometry)*.

10.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: The vertex of the cone is (α, β, γ) and the guiding curve is the conic $z = 0$, $ax^2 + by^2 = 1$ ... ...(1)

Any line through the vertex (α, β, γ) is:

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{... ... ...(2)}$$

It intersects the plane $z = 0$ where, $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$

$$\Rightarrow x - \alpha = -\frac{l\gamma}{n},\ y - \beta = -\frac{m\gamma}{n}$$

$$\Rightarrow x = \alpha - \frac{l\gamma}{n},\ y = \beta - \frac{m\gamma}{n}.$$

Thus the line (2) intersects the plane $z = 0$ at the point $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$. In order that the line (2) may intersect the curve (1), the point $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$ must lie on it ; so we must have,

$$a\left(\alpha - \frac{l\gamma}{n}\right)^2 + b\left(\beta - \frac{m\gamma}{n}\right)^2 = 1 \quad \text{... ... ...(3)}$$

So the variable line (2) will be a generator of the required cone if the condition (3) is satisfied.Eliminating l, m and n from (2) and (3), we shall obtain the equation of the cone.

From (2), we have $\frac{l}{n} = \frac{x-\alpha}{z-\gamma}$ and $\frac{m}{n} = \frac{y-\beta}{z-\gamma}$;

substituting these expressions for $\frac{l}{n}$ and $\frac{m}{n}$ in(3), we get,

$$a\left\{\alpha - \frac{\gamma(x-\alpha)}{z-\gamma}\right\}^2 + b\left\{\beta - \frac{\gamma(y-\beta)}{z-\gamma}\right\}^2 = 1$$

i.e., $a\left(\frac{\alpha z - \gamma x}{z-\gamma}\right)^2 + b\left(\frac{\beta z - \gamma y}{z-\gamma}\right)^2 = 1$

i.e., $a(\alpha z - \gamma x)^2 + b(\beta z - \gamma y)^2 = (z-\gamma)^2$

This is the required equation of the cone.

Ans. to Q. No. 2: A variable line through the vertex (α, β, γ) of the cone is:

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{... ... ...(1)}$$

It intersects the plane y = 0 at the point where,

$$\frac{x-\alpha}{l} = -\frac{\beta}{m} = \frac{z-\gamma}{n}$$

$$\Rightarrow x - \alpha = -\frac{l\beta}{m},\ z - \gamma = -\frac{n\beta}{m}$$

$$\Rightarrow x = \alpha - \frac{l\beta}{m},\ z = \gamma - \frac{n\beta}{m}$$

Thus the line (1) meets the plane y = 0 at the point

$$\left(\alpha - \frac{l\beta}{m}, 0, \gamma - \frac{n\beta}{m}\right).$$

If this line intersects the curve y = 0, $x^2 = 4az$

We must have, $\left(\alpha - \frac{l\beta}{m}\right)^2 = 4a\left(\gamma - \frac{n\beta}{m}\right)$ ... ...(2)

Eliminating l, m and n from (1) and (2), we shall get the equation of the cone whose vertex is (α, β, γ) and base is the parabola y = 0, $x^2 = 4az$.

Now from (1), we have, $\frac{l}{m} = \frac{x-\alpha}{y-\beta}$ and $\frac{n}{m} = \frac{z-\gamma}{y-\beta}$;

Substituting these in (2), we get,

$$\left\{\alpha - \left(\frac{x-\alpha}{y-\beta}\right)\beta\right\}^2 = 4a\left\{\gamma - \left(\frac{z-\gamma}{y-\beta}\right)\beta\right\}$$

i.e., $\left\{\frac{\alpha y - \beta x}{y-\beta}\right\}^2 = 4a\left\{\frac{\gamma y - \beta z}{y-\beta}\right\}$

i.e., $(\alpha y - \beta x)^2 = 4a(y - \beta)(\gamma y - \beta z)$

Which is the required equation of the cone.

Ans. to Q. No. 3: If a right circular cone has vertex at the origin, its axis is the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and its semi-vertical angle is θ, then we know that its equation is: $(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2)\cos^2\theta$.

Hence the required equation of the right circular cone, whoseaxis is $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and semi-vertical angle is 45^0, is:

$(2x + 3y + 4z)^2 = (2^2 + 3^2 + 4^2)(x^2 + y^2 + z^2)\cos^2 45^0$

i.e., $(2x + 3y + 4z)^2 = 29(x^2 + y^2 + z^2)\left(\frac{1}{\sqrt{2}}\right)^2$

i.e., $2(4x^2 + 9y^2 + 16z^2 + 12xy + 24yz + 16zx) = 29(x^2 + y^2 + z^2)$

i.e., $29x^2 + 29y^2 + 29z^2 - 8x^2 - 18y^2 - 32z^2 - 24xy - 48yz - 32zx = 0$

i.e., $21x^2 + 11y^2 - 3z^2 - 24xy - 48yz - 32zx = 0$

Ans. to Q. No. 4: If (α, β, γ) is the vertex, the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ is the axis and θ is the semi-vertical angle of a right circular cone, then its equation is:

$[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2 = (l^2 + m^2 + n^2)\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}\cos^2\theta$

For the given right circular cone vertex is (1, 0, –1), the axis is the line $\frac{x-1}{2} = \frac{y}{-1} = \frac{z+1}{3}$ and the semi-vertical angle is 30^0; hence the required equation of the cone is:

$[2(x-1) + (-1)y + 3(z+1)]^2 = \{2^2 + (-1)^2 + 3^2\}\{(x-1)^2 + y^2 + (z+1)^2\}\cos^2 30^0$

i.e., $(2x - y + 3z + 1)^2 = 14\{(x-1)^2 + y^2 + (z+1)\}\left(\frac{\sqrt{3}}{2}\right)^2$

i.e., $2(2x - y + 3z + 1)^2 = 21\{(x-1)^2 + y^2 + (z+1)^2\}$

Ans. to Q. No. 5: i) We have to find the cone whose vertex is at the origin and which passes through the curve represented by the pair of equations:

$ax^2 + by^2 = 1$ ... ...(1)

and $z = k$ ... ...(2)

We write (2) as $\frac{z}{k} = 1$ and with the help of it we make (1) homogeneous;

thus we get, $ax^2 + by^2 = \left(\frac{z}{k}\right)^2$

i.e., $k^2(ax^2 + by^2) = z^2$

This is the required equation of the cone.

ii) We have to find the cone whose vertex is the origin and which passes through the curve given by

$8yz + 7zx - 3y + 2 = 0$ ... ...(1)

$3z - 2y - 5 = 0$ ... ...(2)

We rewrite (2) as: $3z - 2y = 0$

or $\frac{3z-2y}{5} = 1$ ... ...(3)

and make the equation (1) homogeneous with the help of (3); thereby we get,

$$8yz + 7zx - 3y\left(\frac{3z-2y}{5}\right) + 2\left(\frac{3z-2y}{5}\right)^2 = 0$$

i.e., $200yz + 175zx - 15y(3z - 2y) + 2(3z - 2y)^2 = 0$

i.e., $200yz + 175zx - 45yz + 30y^2 + 18z^2 - 24yz + 8y^2 = 0$

i.e., $38y^2 + 18z^2 - 131yz + 175zx = 0$

This is the required cone.

Ans. to Q. No. 6: i)

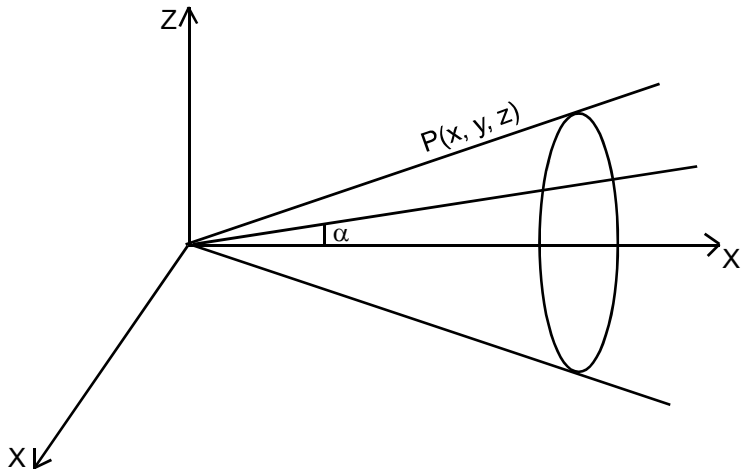


Fig. 10.4

Let P(x, y, z) be any point on the right circular cone. The line OP is a generator of the cone and it has direction ratios x, y, z. Line OP being a generator of the cone, it makes an angle of 45^0 with OY, the axis of the cone. But OY has direction cosines 0, 1, 0. So we get,

$$\cos 45^0 = \frac{0.x + 1.y + 0.z}{\sqrt{x^2 + y^2 + z^2}\sqrt{0^2 + 1^2 + 0^2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\Rightarrow \frac{1}{2} = \frac{y^2}{x^2 + y^2 + z^2}$$

$\Rightarrow x^2 + y^2 + z^2 = 2y^2$

$\Rightarrow x^2 + z^2 = y^2$

Hence the required equation of the right circular cone is $x^2 + z^2 = y^2$.

ii)

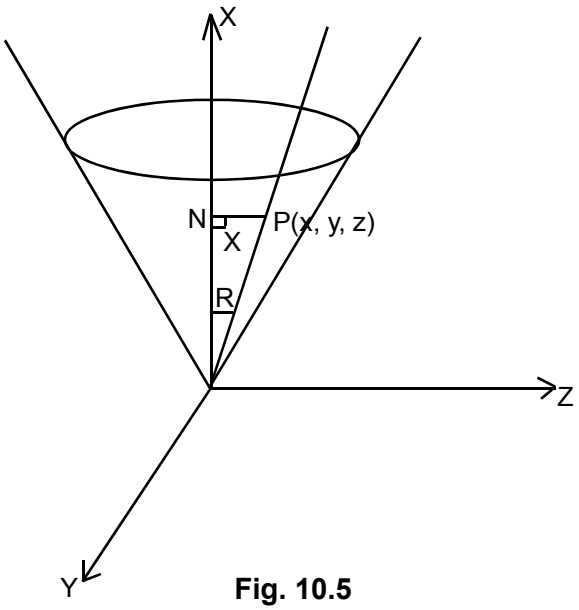


Fig. 10.5

Let P(x, y, z) be any point on the right circular cone whose vertex is at the origin O, axis is OZ and semi-vertical angle is α. Then the line OP is a generator of the cone and so makes angle α with OZ.

Now OP has direction ratios x, y, z; So its direction cosines are:

$$\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

So we get, $\frac{z}{\sqrt{x^2 + y^2 + z^2}} = \cos\alpha$

$$\Rightarrow x^2 + y^2 + z^2 = \frac{z^2}{\cos^2\alpha} = z^2\sec^2\alpha$$

$\Rightarrow x^2 + y^2 = z^2(\sec^2\alpha - 1) = z^2\tan^2\alpha$

As this holds for every point P on the cone,the required equation of the cone is $x^2 + y^2 = z^2\tan^2\alpha$.



10.7 MODEL QUESTIONS

Q.1: Find the equation of the cones with vertex at the origin and which pass through the curves given by:

i) $x^2 + y^2 = 4, z = 2$

ii) $x^2 + y^2 + z^2 = 9, x + y + z = 1.$

Q.2: Find the equation of the right circular cone whose vertex is (1, –1, 2), axis the line $\frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{-2}$ and semi-verticl angle 45^0.

Q.3: Find the equation of the cones whose vertex is at the point (3, 1, 2) and whose guiding curve is the ellipse $2x^2 + 3y^2 = 1, z = 0$.

Q.4: Find the equation of the cones whose vertex is (α, β, γ) and base $y^2 = 4ax, z = 0$.

Q.5: Find the equation of the cones whose vertex is at the point (0, 0, 5) and whose guiding curve is $x^2 + 2xy + 3y^2 = 1, z = 3$.

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UNIT 11: CYLINDER

UNIT STRUCTURE

11.1 Learning Objectives

11.2 Introduction

11.3 Cylinder

11.4 Let Us Sum Up

11.5 Further Reading

11.6 Answers to Check Your Progress

11.7 Model Questions

11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know about cylinder and its equation.

11.2 INTRODUCTION

In unit 9 and 10, we discussed about sphere and cone. In this unit, we will discuss about cylinder and its equation.

11.3 CYLINDER

Definition: A cylinder is a surface generated by a line which moves always remaining parallel to itself and intersects a given curve. The moving line is called a generator and the given curve is called the guiding curve of the cylinder.Each particular position of the moving line is also called a generator.

A cylinder whose guiding curve is a circle and the generators are perpendicular to the plane of the guiding circle is called a right circular cylinder. The normal to the plane of the guiding circle through its centre is called the axis of the right circular cylinder.Any normal section of such a cylinder, i.e., any section by a plane perpendicular to the generators (and so perpendicular to the axis) of the cylinder is a circle,parallel and congruent

to the guiding circle. The radius of any normal section,which is the same as the radius of the guiding circle,is called the radius of the right circular cylinder; clearly it is the perpendicular distance of any point on the cylinder from its axis.

Surfaces represented by the equations of the forms f(x, y) = 0, f(y, z) = 0, f(z, x) = 0: Consider the equation f(x, y) = 0 in which z is absent. This equation is satisfied by the coordinates of all points of the curve in the xy-plane whose two dimensional equation is f(x, y) = 0. Let P, any point on this curve, have coordinates $(x_0, y_0, 0)$ so that we have $f(x_0, y_0) = 0$. Through $P(x_0, y_0, 0)$ draw a line parallel to OZ and let Q be any point on it. Then the coordinates of Q are (x_0, y_0, z) where, PQ = z; clearly z varies as Q moves on this line.

As $f(x_0, y_0) = 0$, the coordinates (x_0, y_0, z) of Q satisfy the equation f(x, y) = 0 whatever be the value of z.

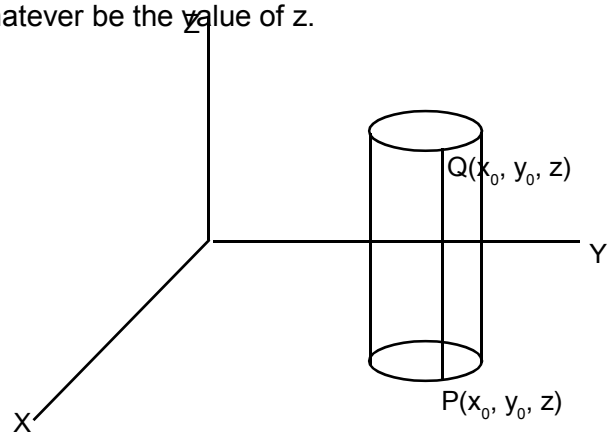


Fig. 11.1

Therefore the coordinates of every point on the line PQ satisfy f(x, y) = 0 and so every point on PQ lies on the surface represented by f(x, y) = 0. In other words the entire line PQ lies on the surface represented by f(x, y) = 0. But P is any point of the curve whose two dimensional equation is f(x, y) = 0. Therefore the locus of the equation f(x, y) = 0 is the cylinder generated by straight lines drawn parallel to the z-axis through all the points of the curve in the z-plane whose two dimensional equation is f(x, y) = 0.

Thus an equation of the form f(x, y) = 0, which does not contain z, represents a cylinder whose generators are parallel to z-axis and whose guiding curve is given by z = 0, f(x, y) = 0. Similarly f(y, z) = 0 and f(z, x) = 0

respectively represent cylinders whose generators are parallel to x-axis and y-axis respectively.

Equation of the cylinder generated by lines parallel to a fixed line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and having the conic z = 0, $ax^2 + by^2 = 1$ as its guiding curve: Let (α, β, γ) be any point on the cylinder. Through this point there is a generator of the cylinder which is parallel to the given line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

The equations to this generator are:

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{... ... ...(1)}$$

As this is a generator it intersects the guiding curve given by

$z = 0, ax^2 + by^2 = 1$ ... ...(2)

Putting z = 0 in (1) we get, $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=-\frac{\gamma}{n}$

$$\Rightarrow x=\alpha-\frac{l\gamma}{n},\ y=\beta-\frac{m\gamma}{n}.$$

So the line meets the plane z = 0 at the point $\left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right)$, and as this line intersects the curve (2), we must have,

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+b\left(\beta-\frac{m\gamma}{n}\right)^2=1.$$

Thus the coordinates (α, β, γ) of any point on the cylinder satisfy the equation:

$$a\left(x-\frac{lz}{n}\right)^2+b\left(y-\frac{mz}{n}\right)^2=1$$

i.e., $a(nx - lz)^2 + b(ny - mz)^2 = n^2$

Hence the required equation of the cylinder is:

$a(nx - lz)^2 + b(ny - mz)^2 = n^2$

Equation of the right circular cylinder whose axis and radius are given: Suppose that a right circular cylinder has radius r and its axis is the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$; To find its equation.

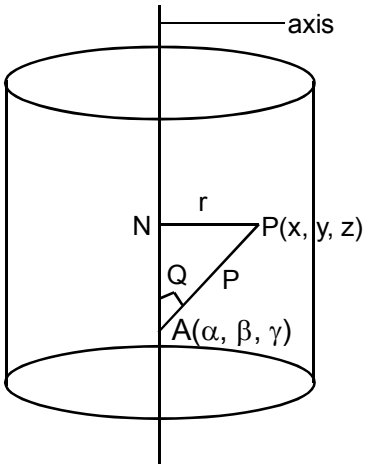


Fig. 11.2

A(α, β, γ) is a point on the axis. Let P(x, y, z) be any point on the right circular cylinder. Draw PN perpendicular to the axis; then PN = r, r being the radius of the right circular cylinder. If AP = ρ, the direction cosines of the line AP are $\frac{x-\alpha}{\rho}, \frac{y-\beta}{\rho}, \frac{z-\gamma}{\rho}$. Also,as given, l, m, n are direction ratios of the line AN (axis of the cylinder) so that AN has direction cosines $\frac{l}{k}, \frac{m}{k}, \frac{n}{k}$ where, $k^2 = l^2 + m^2 + n^2$.

So, if θ be the angle between AP and AN, we have,

$$\cos\theta = \frac{x-\alpha}{\rho}.\frac{l}{k} + \frac{y-\beta}{\rho}.\frac{m}{k} + \frac{z-\gamma}{\rho}.\frac{n}{k}$$

$$= \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\rho k}$$

From ΔAPN right-angled at N, we get,

$PN = AP\sin\theta$

$\Rightarrow r = \rho\sin\theta$

$\Rightarrow r^2 = \rho^2\sin^2\theta = \rho^2(1 - \cos^2\theta)$

$$\Rightarrow r^2 = \rho^2 - \rho^2\left[\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\rho k}\right]^2$$

$$\Rightarrow r^2 = \rho^2 - \frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2}{k^2}$$

$$\Rightarrow (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - \frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2}{l^2+m^2+n^2} = r^2$$

as: $\rho^2 = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2$ and $k^2 = l^2 + m^2 + n^2$

As this is satisfied by the coordinates (x, y, z) of any point P on the cylinder, it follows that the equation of the given right circular cylinder is:

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 - \frac{[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2}{l^2 + m^2 + n^2} . = r^2$$

Corollary: The equation of the given right circular cylinder of radius r whose axis is the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ (which passes through the origin) is:

$$x^2 + y^2 + z^2 - \frac{(lx + my + nz)^2}{l^2 + m^2 + n^2} = r^2$$

i.e., $(l^2 + m^2 + n^2)(x^2 + y^2 + z^2) - (lx + my + nz)^2 = (l^2 + m^2 + n^2)r^2$

i.e., $(l^2 + m^2 + n^2)(x^2 + y^2 + z^2 - r^2) = (lx + my + nz)^2$.

ILLUSTRATIVE EXAMPLES:

Example 1: Find the equation of the cylinder generated by the lines parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{5}$, the guiding curve being the parabola x = 0, $y^2 = 8z$.

Solution: Let (α, β, γ) be any point on the cylinder. The equations to its generator through (α, β, γ), which is parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{5}$, are:

$$\frac{x - \alpha}{1} = \frac{y - \beta}{-2} = \frac{z - \gamma}{5} \qquad \text{... ... ...(1)}$$

This generator intersects the plane x = 0 at the point where, x = 0 and $\frac{-\alpha}{1} = \frac{y - \beta}{-2} = \frac{z - \gamma}{5}$ giving $y = \beta + 2\alpha$ and $z = \gamma - 5\alpha$.

Thus the generator (1) intersects the plane x = 0 at $(0, \beta + 2\alpha, \gamma - 5\alpha)$. As this generator must intersect the base curve x = 0, $y^2 = 8z$, this point must lie on it; hence, $(\beta + 2\alpha)^2 = 8(\gamma - 5\alpha)$

As (α, β, γ) is any point on the cylinder, it follows that the required equation of the cylinder is $(y + 2x)^2 = 8(z - 5x)$.

Example 2: Find the equation of the right circular cylinder of radius 5 if its axis passes through the point (3,1,5) and has direction ratios 2, –1, 2.

Solution: If a right circular cylinder of radius r has the line,

$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$ as its axis, then the equation of the cylinder is:

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - \frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2}{l^2+m^2+n^2} = r^2$$

Hence the axis of the right circular cylinder is the line

$\frac{x-3}{2} = \frac{y-1}{-1} = \frac{z-5}{2}$ and its radius is 5; hence the equation of the right circular cylinder is:

$$(x-3)^2 + (y-1)^2 + (z-5)^2 - \frac{[2(x-3)+(-1)(y-1)+2(z-5)]^2}{2^2+(-1)^2+2^2} = 5^2$$

i.e., $9[(x-3)^2 + (y-1)^2 + (z-5)^2] - (2x - y + 2z - 15)^2 = 225$

i.e., $9(x^2 + y^2 + z^2 - 6x - 2y - 10z + 35) -$

$(4x^2 + y^2 + 4z^2 + 225 - 4xy + 8zx - 60x - 4yz + 30y - 60z) = 225$

i.e., $5x^2 + 8y^2 + 5z^2 + 4yz + 4xy + 6x - 48y - 30z - 135 = 0$

Example 3: Find the equation of the right circular cylinder of radius $\sqrt{7}$ whose axis is the line $\frac{x}{1} = \frac{y}{0} = \frac{z}{-2}$.

Solution: We know that the equation of a right circular cylinder, whose radius is r and whose axis is the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, is:

$(l^2 + m^2 + n^2)(x^2 + y^2 + z^2 - r^2) = (lx + my + nz)^2$.

Here the radius is $\sqrt{7}$ and the axis is the line $\frac{x}{1} = \frac{y}{0} = \frac{z}{-2}$;

Hence the required equation of the right circular cylinder is:

$\{1^2 + 0^2 + (-2)^2\}\{x^2 + y^2 + z^2 - (\sqrt{7})\} - \{1.x + 0.y + (-2).z\}^2$

i.e., $5(x^2 + y^2 + z^2 - 7) = (x - 2z)^2 = x^2 + 4z^2 - 4zx$

i.e., $4x^2 + 5y^2 + z^2 + 4zx - 35 = 0$.

Example 4: Find the equation of the cylinder whose generators are parallel to the x-axis and which passes through the curve of intersection of the plane $2x - 3y + z = 1$ and the surface $3y^2 - 5z^2 = 12x$.

[Observe that the generators are parallel to the x-axis and so the equation of the cylinder will be of the form f(y, z) = 0 which is free from x.]

Solution: The generators of the cylinder pass through the curve of intersection of the plane $2x - 3y + z = 1$ ... ...(1)

and the surface $3y^2 - 5z^2 = 12x$ ... ...(2)

We eliminate x from (1) and (2); from (1) we have, $x = \frac{1+3y-z}{2}$

and putting this in (2), we get,

$$3y^2 - 5z^2 = 12\left(\frac{1+3y-z}{2}\right) = 6(1 + 3y - z)$$

i.e., $3y^2 - 5z^2 - 18y + 6z - 6 = 0$ ... ...(3)

(3) is an equation without x and so it represents a cylinder whose generators are parallel to x-axis.Moreover it is satisfied whenever the equations (1) and (2) are simultaneously satisfied. In other words (3) is satisfied by the coordinates of every point of the curve of intersection of the plane (1) and the surface (2).

In other words the cylinder (3), whose generators are parallel to the x-axis, passes through the said curve and hence it is the required cylinder.

Remarks: In the above solution we have given reasons in detail so that the learner can understand the principle involved. However we can shorten the solution as shown in the following example.

Example 5: Find the equation of the cylinder generated by the lines parallel to the z-axis and passing through the curve of intersection of the plane $lx + my + nz = p$ and the surface $ax^2 + by^2 + cz^2 = 1$.

Solution: The cylinder is generated by the lines parallel to the z-axis and it passes through the curve of intersection of the plane:

$$lx + my + nz = p \quad \text{... ... ...(1)}$$

and the surface $ax^2 + by^2 + cz^2 = 1$ ... ...(2)

The equation of the cylinder will be obtained by eliminating z from (1) and (2).

From (1) we have, $z = \frac{p - lx - my}{n}$

and putting this for z in (2), we get,

$$ax^2 + by^2 + c\left(\frac{p - lx - my}{n}\right)^2 = 1$$

Which is the required equation of the cylinder.



CHECK YOUR PROGRESS

Q.1: Find the equation of the cylinder whose generators are parallel to the y-axis and intersect the curve given by $4x + y - z = 2$ and $x^2 - 3y^2 + 2z^2 = 5$.

Q.2: A right circular cylinder of radius a has the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ as its axis. Find its equation.

Q.3: The axis of a right circular cylinder passes through the point (1, –2, 0) and has direction ratios 2, 3, 6. If the radius of the right circular cylinder is 5, Find its equation.



11.4 LET US SUM UP

- The equation of the right circular cylinder whose radius is r and axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ is:

  $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - \frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2}{l^2+m^2+n^2} = r^2$

- The equation of the right circular cylinder of radius r whose axis is the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ (which passes through the origin) is:

  $(l^2 + m^2 + n^2)(x^2 + y^2 + z^2 - r^2) = (lx + my + nz)^2$.



11.5 FURTHER READING

1) Jain, P. K. & Ahmad, Khalil. *Textbook of Analytical Geometry of Three Dimensions*. New Delhi: New Age International (P) Limited.
2) Khan, R. M. *Analytical Geometry of Two & Three Dimensions & Vector Analysis*. New Central Book Agency (P) Limited.
3) Kishan, Hari. *Coordinate Geometry of Two Dimensions*. New Delhi: Atlantic Publishers & Distributors.

4) Pal, Sat & Pal, Harbans. *A Textbook of Two Dimensional Geometry*. New Age International Publishers.

5) Sharma, G. C. & Jain, Madhu. *Coordinate Geometry (2D and 3D)*. Galgotia Publications Pvt Lid.

6) Sharma, A. K. *Coordinate Geometry (2D)*. New Delhi: Discovery Publishing House.

7) Vasistha, A. R. & Agrawal, *D. C. Analytical Geometry of Three Dimensions (Coordinate of Solid Geometry)*.

11.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: We have to find the equation of the cylinder whose generators are parallel to the y-axis and intersect the curve given by

$4x + y - z = 2$ ... ...(1)

and $x^2 - 3y^2 + 2z^2 = 5$ ... ...(2)

This equation will be obtained by eliminating y from the equations(1) and (2).

From (1) $y = 2 + z - 4x$ and putting this expression for y in (2), we get,

$x^2 - 3(2 + z - 4x)^2 + 2z^2 = 5$

$\Rightarrow x^2 - 3(4 + z^2 + 16x^2 + 4z - 16x - 8zx) + 2z^2 = 5$

$\Rightarrow -47x^2 - z^2 + 24zx + 48x - 12z - 17 = 0$

$\Rightarrow 47x^2 + z^2 - 24zx - 48x + 12z + 17 = 0$

Which is the required equation of the cylinder.

Ans. to Q. No. 2:

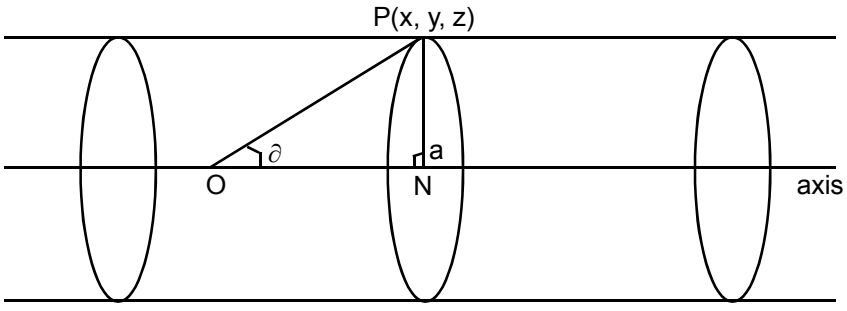


Fig. 11.3

Let P(x, y, z) be any point on the right circular cylinder of radius a whose axis is the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. The axis passes through the origin.

If PN is dropped perpendicular from P to the axis of the cylinder, we have PN = radius of the cylinder = a.

Let $\angle PON = \theta$; clearly from the figure,

$$\cos\theta = \frac{ON}{OP} = \frac{\sqrt{OP^2 - PN^2}}{OP}$$

and so, $\cos\theta = \frac{\sqrt{x^2 + y^2 + z^2 - a^2}}{\sqrt{x^2 + y^2 + z^2}}$

But OP has direction ratios x, y, z and ON (the axis) has direction ratios l, m, n. As θ is the angle between OP and ON, we have,

$$\cos\theta = \frac{lx + my + nz}{\sqrt{l^2 + m^2 + n^2}\sqrt{x^2 + y^2 + z^2}}$$

Hence we get, $\frac{lx + my + nz}{\sqrt{l^2 + m^2 + n^2}\sqrt{x^2 + y^2 + z^2}}$

$$= \frac{\sqrt{x^2 + y^2 + z^2 - a^2}}{\sqrt{x^2 + y^2 + z^2}}$$

Whence, on squaring and multiplying crosswise, we get,

$(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2 - a^2)$

This is the required equation of the right circular cylinder.

Ans. to Q. No. 3: The axis of the right circular cylinder is the line,

$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{6}$ and its radius is 5.

So the equation of the right circular cylinder is:

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - \frac{[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2}{l^2 + m^2 + n^2} = r^2$$

(formula in usual symbols)

i.e., $(x-1)^2 + (y+2)^2 + z^2 - \frac{[2(x-1) + 3(y+2) + 6z]^2}{2^2 + 3^2 + 6^2} = 5^2$

i.e., $49[(x-1)^2 + (y+2)^2 + z^2] - (2x + 3y + 6z + 4)^2 = 1225$

This is the required equation of the right circular cone.



11.7 MODEL QUESTIONS

Q.1: Find the equation of the cylinder whose generators are parallel to the line $x = \frac{-y}{2} = \frac{z}{3}$ and whose guiding curve is the ellipse $x^2 + 2y^2 = 1$, $z = 3$.

Q.2: Find the equation of the right circular cylinderof radius of radius 2 whose axis passes through (12, 2, 3) and has direction cosines proportional to 2, –3, 6.

Q.3: Find the equation of the right circular cylinder of radius 3 and axis $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$.

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UNIT 12: INTRODUCTION TO VECTORS

UNIT STUCTURE

12.1 Learning Objectives

12.2 Introduction

12.3 Vector and Scalar Quantity

12.4 Types of Vectors

12.5 Operation on Vectors

12.5.1 Additions of Vectors

12.5.2 Subtraction of Vectors

12.5.3 Multiplication of Vectors by Scalars

12.6 Section Formula

12.7 Components of a Vector

12.8 Let Us Sum Up

12.9 Further Reading

12.10 Answers to check Your Progress

12.11 Model Questions

12.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- know the terms scalar quantity and vector quantity
- classify different types of vectors
- define position vector of a point;
- find direction cosines and direction ratios of a vector
- find addition of vectors
- find difference of two vectors
- multiply a vector by a scalar
- write a vector in component form
- know section formula.

12.2 INTRODUCTION

In mathematics, physics and engineering, we frequently come across with two types of quantities, namely scalar quantities and vector quantities.

In this unit, we will discuss vector and its different types. We will also discuss different operations on vectors.

12.3 VECTOR AND SCALAR QUANTITY

We define scalar and vector quantities with examples.

Scalar:

Definition: The quantities which have magnitude only are called scalar quantities

Examples: mass, volume, area, density, temperature etc.

Vector:

Definition: The quanities which have both magnitude and direction are called vector quantities.

Examples: Displacement, velocity, acceleration, momentum, weight, force etc.

Representation of Vectors: A vector is represented by a directed line segment such that the length of the line segment is the magnitude of the vector and the direction of arrow marked at one end emphasizes the direction of the vector.

A vector whose magnitude is the length PQ and whose direction is from P to Q is denoted by $\vec{PQ}$. The point P is called the initial point and Q is called the terminal point of $\vec{PQ}$.

Note: 1) Vectors are denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$ etc.

2) The magnitude of $\vec{PQ}$ is denoted by $|\vec{PQ}|$ or PQ.

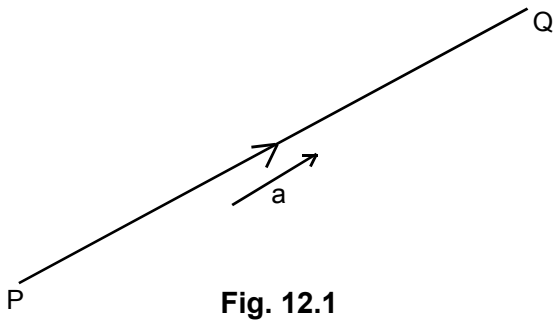


Fig. 12.1

3) The sense of $\vec{AB}$ is from A to B and that of $\vec{BA}$ is from B to A.

Thus the sense of a directed line segment is from its initial point to the terminal point.

Position Vector: Consider a point P(x, y, z) in space with respect to the origin O(0, 0, 0). Then, the vector $\vec{OP}$ with O and P as it initial and terminal points is called the position vector of P with respect to O.

The magnitude of $\vec{OP}$ (or $\vec{r}$) is given by $|\vec{OP}| = \sqrt{x^2 + y^2 + z^2}$.

12.4 TYPES OF VECTORS

i) **Zero or Null Vector:** A vector whose magnitude is zero and direction is indeterminate is called the zero or null vector.It is denoted by $\vec{0}$. In this case, the initial and terminal points coincide. Thus, $\vec{AA}$, $\vec{BB}$ etc. are zero vectors.Vectors other than the null vector are called proper vectors.

ii) **Unit Vector:** A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. The unit vector in the direction of a given vector $\vec{a}$ is denoted by $\hat{a}$ and defined as, $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ i.e., $\vec{a} = |\vec{a}|\hat{a}$

i.e., Vector = (Magnitude)(Unit Vector)

Similarly, $\hat{b} = \dfrac{\vec{b}}{|\vec{b}|}$, $\hat{c} = \dfrac{\vec{c}}{|\vec{c}|}$ and so on.

iii) **Co-initial Vectors:** Vectors having the same initial point are called co-initial vec tors.

iv) **Co-terminus Vectors:** Vectors having the same terminal point are called co-terminus vectors.

v) **Collinear or Parallel Vectors:** Vectors are said to be collinear or parallel if they have the same line of action or have the lines of action parallel to one another.

vi) **Coplanar Vectors:** Vectors are said to be coplanar if they are parallel to the same plane or they lie in the same plane.

vii) **Negative Vector:** The vector which has the same magnitude as that of $\vec{a}$ but opposite direction is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$. Thus if $\vec{AB} = \vec{a}$ then, $\vec{BA} = -\vec{a}$.
Thus, $\vec{AB} = -\vec{BA} = \vec{PQ} = -\vec{QP}$ and so on.

viii) Equal vectors: Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, written as $\vec{a} = \vec{b}$, if they have the (i) same magnitude (ii) same direction.

Example 1: State which of the following are scalars and which are vectors. Give reasons: (a) time period, (b) distance, (c) velocity, (d) Mass, (e) Temperature, (f) Force.

Solution:

a) Time period is a scalar quantity as it has only magnitude.

b) Distance is a scalar quantity as it has only magnitude.

c) Velocity is a vector quantity as it has both magnitude as well as direction.

d) Mass is a scalar quantity as it has only magnitude.

e) Temperature is a scalar quantity as it has only magnitude.

f) Force is a vector quantity as it has both magnitude as well as direction.

Example 2: Represent graphically:

a) a force 50N in a direction 60^0 north of east.

b) a displacement of 30km in a direction 30^0 north of west.

Solution:

(a)

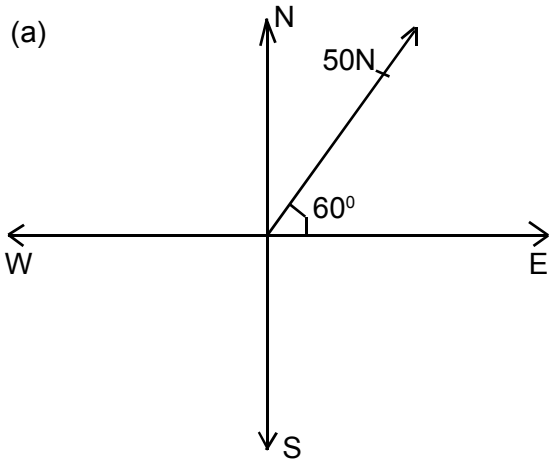


Fig 12.2

(b)

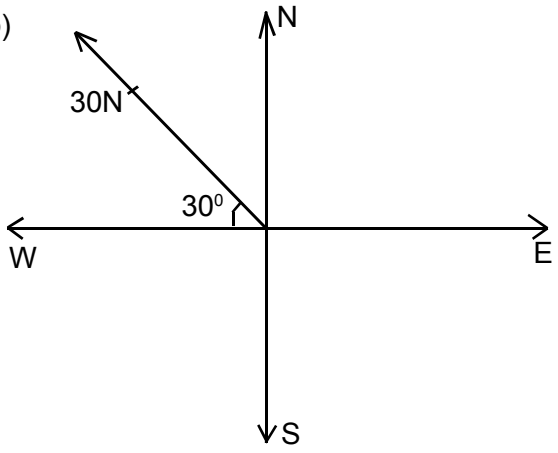


Fig 12.3



CHECK YOUR PROGRESS

Q.1: State which of the following are scalars and which are vectors. Give reasons.

a) Density b) Weight c) Momentum

d) Work done e) volume f) Acceleration

Q.2: Represent graphically:

a) a force 100N in a direction 45^0 west of north.

b) a displacement 50 km in a direction 60^0 south of east.

12.5 OPERATION ON VECTORS

12.5.1 Addition of Vectors

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ be the two given vectors. Now, we join OB.

Then $\vec{OB}$ represents the addition (sum) of the vectors $\vec{a}$ and $\vec{b}$.

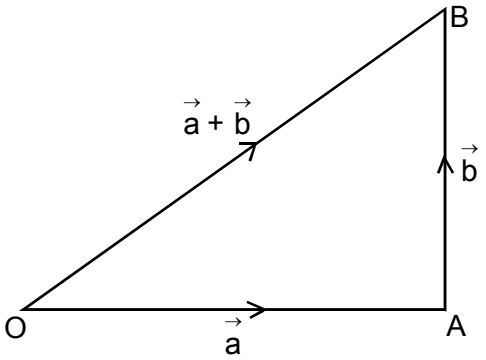


Fig. 12.4

This is written as $\vec{OA} + \vec{AB} = \vec{OB}$

Thus, $\vec{OB} = \vec{OA} + \vec{AB} = \vec{a} + \vec{b}$.

This is known as the triangle law of addition of vectors.

Applying the triangle law of addition of vectors in ΔABC, we have,

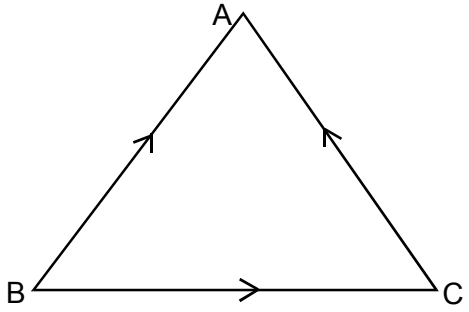


Fig. 12.5

$\vec{BC} + \vec{CA} = \vec{BA}$

or, $\vec{BC} + \vec{CA} = -\vec{AB}$

or, $\vec{BC} + \vec{CA} + \vec{AB} = \vec{0}$

Parallelogram law of Addition of Vectors: If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two adjacent sides of a parallelogram, then their sum $\vec{c}$ is represented by the diagonal of the parallelogram as shown in the figure.

We have, $\vec{OP} + \vec{PR} = \vec{OR}$

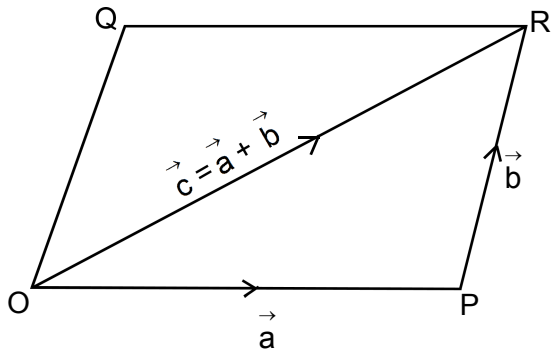


Fig. 12.6

Properties of Addition of Vectors:

Property 1: Vector addition is commutative i.e., if $\vec{a}$ and $\vec{b}$ are any two vectors then, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

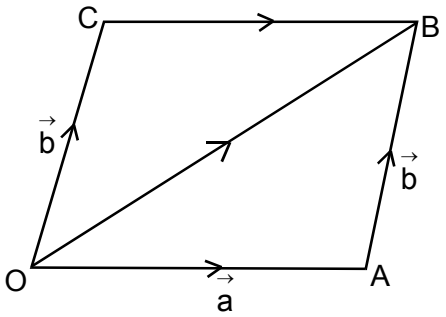


Fig. 12.7

Let, $\vec{a} = \vec{OA}$, $\vec{b} = \vec{AB}$

In triangle ΔOAB, $\vec{OA} + \vec{AB} = \vec{OB}$ (by triangle law of addition)

$\Rightarrow \vec{a} + \vec{b} = \vec{OB}$ ... ...(1)

We complete the paralleogram OABC

$\vec{CB} = \vec{OA} = \vec{a}$, $\vec{OC} = \vec{AB} = \vec{b}$

In ΔOCB, we have, $\vec{OC} + \vec{CB} = \vec{OB}$

$\Rightarrow \vec{b} + \vec{a} = \vec{OB}$ ... ...(2)

From (1) and (2), we have, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

$\therefore$ Vector addition is commutative.

Property 2: Vector addition is associative i.e., for any three vectors

$\vec{a}, \vec{b}, \vec{c}$.

$\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$

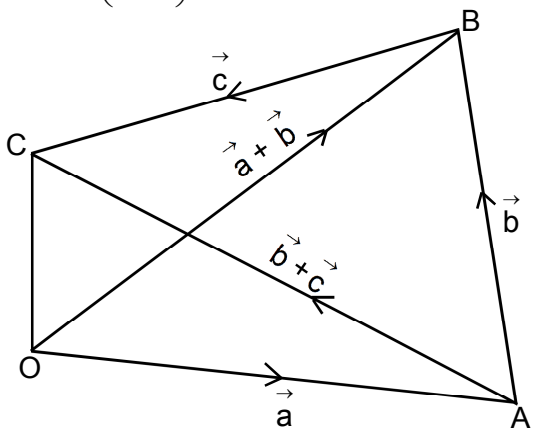


Fig. 12.8

Let, $\vec{OA} = \vec{a}$, $\vec{AB} = \vec{b}$, $\vec{BC} = \vec{c}$

We join O and B, O and C and A and C.

In ΔOAB, $\vec{OA} + \vec{AB} = \vec{OB}$

$\Rightarrow \vec{a} + \vec{b} = \vec{OB}$ ... ...(1)

In ΔOBC, $\vec{OB} + \vec{BC} = \vec{OC}$

$\Rightarrow \left(\vec{a}+\vec{b}\right) + \vec{c} = \vec{OC}$ ... ...(2) [using (1)]

In ΔABC, $\vec{AB} + \vec{BC} = \vec{AC}$

$\Rightarrow \vec{b} + \vec{c} = \vec{AC}$ ... ...(3)

In ΔOAC, $\vec{OA} + \vec{AC} = \vec{OC}$

$\Rightarrow \vec{a} + \left(\vec{b}+\vec{c}\right) = \vec{OC}$ ... ...(4) [using (2)]

From (2) and (4), we have, $\left(\vec{a}+\vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b}+\vec{c}\right)$

$\therefore$ Vector addition is associative.

Property 3: Existence of additive identity: For every vector $\vec{a}$,

$\vec{a} + \vec{0} = \vec{0} + \vec{a}$ where, $\vec{0}$ is the null vector

Let, $\vec{OA} = \vec{a}$

Then, $\vec{a} + \vec{0} = \vec{OA} + \vec{AA} = \vec{OA} = \vec{a}$ $\quad \therefore \vec{a} + \vec{0} = \vec{a}$

Also, $\vec{0} + \vec{a} = \vec{OO} + \vec{OA} = \vec{OA} = \vec{a}$

$\therefore \vec{0} + \vec{a} = \vec{a}$

$\therefore \vec{a} + \vec{0} = \vec{0} + \vec{a}$

Property 4 : Existence of additive inverse: For every vector $\vec{a}$, there corresponds a vector $-\vec{a}$, such that, $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$

Let, $\vec{OA} = \vec{a}$, then, $\vec{OA} = -\vec{a}$

$\therefore \vec{a} + (-\vec{a}) = \vec{OA} + \vec{AO} = \vec{OO} = \vec{O}$

$(-\vec{a}) + \vec{a} = \vec{AO} + \vec{OA} = \vec{AA} = \vec{O}$

Hence, $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$

12.5.2 Subtraction of Vectors

Let $\vec{a}$ and $\vec{b}$ are any two vectors,then the subtraction of $\vec{b}$ from $\vec{a}$ is defined as the vector sum of $\vec{a}$ and $-\vec{b}$ and is denoted by $\vec{a} - \vec{b}$.

$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

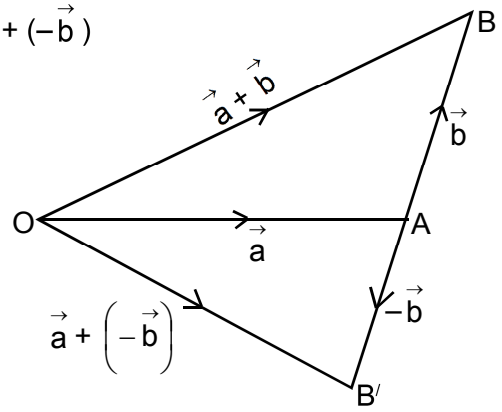


Fig. 12.9

Let, $\vec{a} = \vec{OA}$, $\vec{b} = \vec{AB}$

Then, $\vec{OB} = \vec{OA} + \vec{AB} = \vec{a} + \vec{b}$

We produce BA to B/, such that, AB = AB/

$\therefore \overrightarrow{AB'} = -\overrightarrow{BA} = -\vec{b}$

By Triangle law of vector additon

$\overrightarrow{OB'} = \overrightarrow{OA} + \overrightarrow{AB'} = \vec{a} + (-\vec{b}) = \vec{a} - \vec{b}$

12.5.3 Multiplication of Vectors by Scalars

Let m be a scalar and $\vec{a}$ be a vector, then $m\vec{a}$ is defined as a vector having the same support as that of $\vec{a}$ such that its magnitude is |m| times the magnitude of $\vec{a}$ and its direction is same as or opposite to the direction of $\vec{a}$ according as m is positive or negative.

If $\vec{a}$ is a vector then $5\vec{a}$ is a vector whose magnitude is 5 times the magnitude of $\vec{a}$ and whose direction is same as that of $\vec{a}$. But $-5\vec{a}$ is a vector whose magnitude is 5 times the magnitude of $\vec{a}$ and whose direction is opposite to $\vec{a}$.

Properties of Multiplication of vectors by a scalar

For vectors $\vec{a}$, $\vec{b}$ and scalars m, n we have,

i) $m(-\vec{a}) = (-m)\vec{a} = -(m\vec{a})$ ii) $(-m)(-\vec{a}) = m\vec{a}$

iii) $m(n\vec{a}) = (mn)\vec{a} = n(m\vec{a})$ iv) $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

v) $m(\vec{a} + \vec{b}) = m\vec{a} + n\vec{b}$

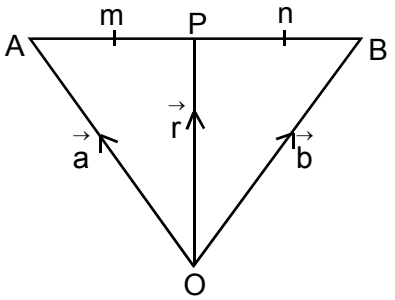


Fig. 12.10

12.6 SECTION FORMULA

Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let P be a point dividing AB internally in the ratio m : n. Then the position vector of P is given by $\overrightarrow{OP} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

Proof: Let O be the origin

Then, $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$

Let the position vector of P with respect to O be $\vec{r}$ i.e., $\vec{OP} = \vec{r}$

Let P divide AB internally in the ratio m : n.

Then, $\frac{AP}{PB} = \frac{m}{n}$

$\Rightarrow nAP = mPB$

$\Rightarrow n\vec{AP} = m\vec{PB}$

$\Rightarrow n(\vec{OP} - \vec{OA}) = m(\vec{OB} - \vec{OP})$

$\Rightarrow n(\vec{r} - \vec{a}) = m(\vec{b} - \vec{r})$

$\Rightarrow n\vec{r} - n\vec{a} = m\vec{b} - m\vec{r}$

$\Rightarrow n\vec{r} + m\vec{r} = m\vec{b} + n\vec{a}$

$\Rightarrow (m + n)\vec{r} = m\vec{b} + n\vec{a}$

$\Rightarrow \vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$

Note: If P is the mid point of AB, then it divides AB in the ratio 1 : 1.

$\vec{OP} = \frac{1.\vec{b} + 1.\vec{a}}{1+1} = \frac{\vec{a} + \vec{b}}{2}$

Section Formula-External Division: Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let P be a point dividing **AB** externally in the ratio m : n.

Then the position vector of P is given by $\vec{OP} = \frac{m\vec{b} - n\vec{a}}{m-n}$

Proof: Let O be the origin

Then, $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$

Let the position vector of P with respect to O be $\vec{r}$ i.e., $\vec{OP} = \vec{r}$

Let P divide AB externally in the ratio m : n

Then, $\frac{AP}{PB} = \frac{m}{n}$

$\Rightarrow nAP = mPB$

$\Rightarrow n\vec{AP} = -m\vec{PB}$ [$\vec{AP}$ and $\vec{PB}$ are in opposite direction]

$\Rightarrow n(\vec{OP} - \vec{OA}) = -m(\vec{OB} - \vec{OP})$

$\Rightarrow n(\vec{r} - \vec{a}) = m(\vec{r} - \vec{b})$

$\Rightarrow m\vec{b} - n\vec{a} = m\vec{r} - n\vec{r}$

$\Rightarrow m\vec{b} - n\vec{a} = (m - n)\vec{r}$

$\Rightarrow \vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$

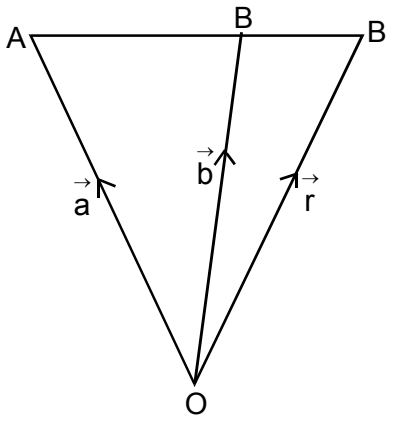


Fig. 12.11

ILLUSTRATIVE EXAMPLES:

Example 1: i) Let P and Q be two points with position vectors $\vec{OP} = 3\vec{a} - 2\vec{b}$ and $\vec{OQ} = \vec{a} + \vec{b}$. Find the position vector of a point R which divides the line joining P and Q in the ratio 2 :1. (i) internally, and (ii) externally.

Solution:

i) $\vec{OR} = \frac{2\vec{OQ} + \vec{OP}}{2+1} = \frac{2\left(\vec{a} + \vec{b}\right) + \left(3\vec{a} - 2\vec{b}\right)}{3} = \frac{5\vec{a}}{3}$

ii) $\vec{OR} = \frac{2\vec{OQ} - \vec{OP}}{2-1} = \frac{2\left(\vec{a} + \vec{b}\right) - \left(3\vec{a} - 2\vec{b}\right)}{1} = -\vec{a} + 4\vec{b}$

12.7 COMPONENTS OF A VECTOR

Vector in the rectangular coordinate system:

Theorem 12.7.1: If P is a point in a two dimensional plane which has coordinates (x, y) then, $\vec{OP} = x\hat{i} + y\hat{j}$, where, $\hat{i}$ and $\hat{j}$ are unit vectors along x-axis and y-axis respectively.

Proof:

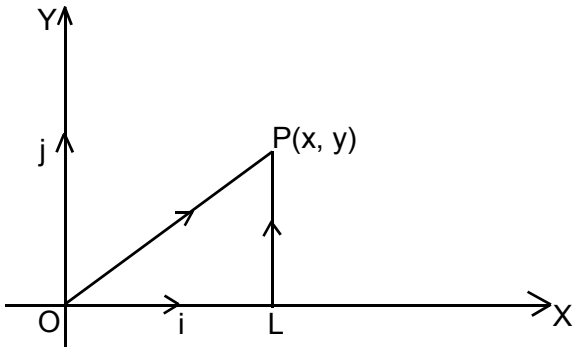


Fig. 12.12

Let P(x, y) be a point in a plane with reference to OX and OY as co-ordinate axes as shown in the figure.

Draw PL perpendicular to OX.

Then OL = x and LP = y

Let $\hat{i}$, $\hat{j}$ be the unit vectors along OX and OY respectively

Then $\vec{OL} = x\hat{i}$ and $\vec{LP} = y\hat{j}$.

Vectors $\vec{OL}$ and $\vec{LP}$ are known as the components of $\vec{OP}$ along x-axis and y-axis respectively.

Now by triangle law of addition

$\vec{OP} = \vec{OL} + \vec{LP} = x\hat{i} + y\hat{j} = \vec{r}$ (say)

$\therefore \vec{r} = x\hat{i} + y\hat{j}$

Now, $OP^2 = OL^2 + LP^2 = x^2 + y^2$

$\Rightarrow OP = \sqrt{x^2 + y^2}$

$\Rightarrow |\vec{r}| = \sqrt{x^2 + y^2}$

Thus, if a point P in a plane has coordinates (x, y) then,

i) $\vec{r} = \vec{OP} = x\hat{i} + y\hat{j}$

ii) $|\vec{r}| = |\vec{OP}| = |x\hat{i} + y\hat{j}| = \sqrt{x^2 + y^2}$

iii) The component of $\vec{OP}$ along x-axis is a vector $x\hat{i}$ and the component of $\vec{OP}$ along y-axis is a vector $y\hat{j}$.

Components of a Vector $\vec{AB}$ in terms of coordinates of A and B

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in XOY plane.Let $\hat{i}$ and $\hat{j}$ be unit vectors along x-axis and y-axis repectively.

$AN = x_2 - x_1$, $BN = y_2 - y_1$

$\therefore \vec{AN} = (x_2 - x_1)\hat{i}$, $\vec{BN} = (y_2 - y_1)\hat{j}$

Now by triangle law of addition,

$\vec{AB} = \vec{AN} + \vec{BN} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$

Component of $\vec{AB}$ along *x*-axis $= (x_2 - x_1)\hat{i}$

Component of $\vec{AB}$ along *y*-axis $= (y_2 - y_1)\hat{j}$

Magnitude of $\vec{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

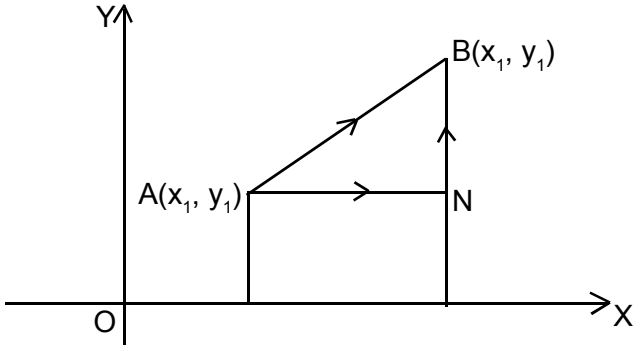


Fig. 12.13

Addition, Subtraction, Multiplication of a vector by a scalar and equality of vectors in terms of components:

Let, $\vec{a} = a_1\hat{i} + a_2\hat{j}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j}$, then,

i) $\vec{a} + \vec{b} = (a_1\hat{i} + a_2\hat{j}) + (b_1\hat{i} + b_2\hat{j}) = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j}$

ii) $\vec{a} - \vec{b} = (a_1\hat{i} + a_2\hat{j}) - (b_1\hat{i} + b_2\hat{j}) = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j}$

iii) $m\vec{a} = m(a_1\hat{i} + a_2\hat{j}) = ma_1\hat{i} + ma_2\hat{j}$, where m is a constant

iv) $\vec{a} = \vec{b} \Rightarrow a_1\hat{i} + a_2\hat{j} = b_1\hat{i} + b_2\hat{j} \Rightarrow a_1 = b_1$ and $a_2 = b_2$.

Theorem 12.7.2: If a point P in space has coordinate (x, y, z) then its position vector $\vec{r}$ is $x\hat{i} + y\hat{j} + z\hat{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$ where, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along OX, OY and OZ repectively.

Proof:

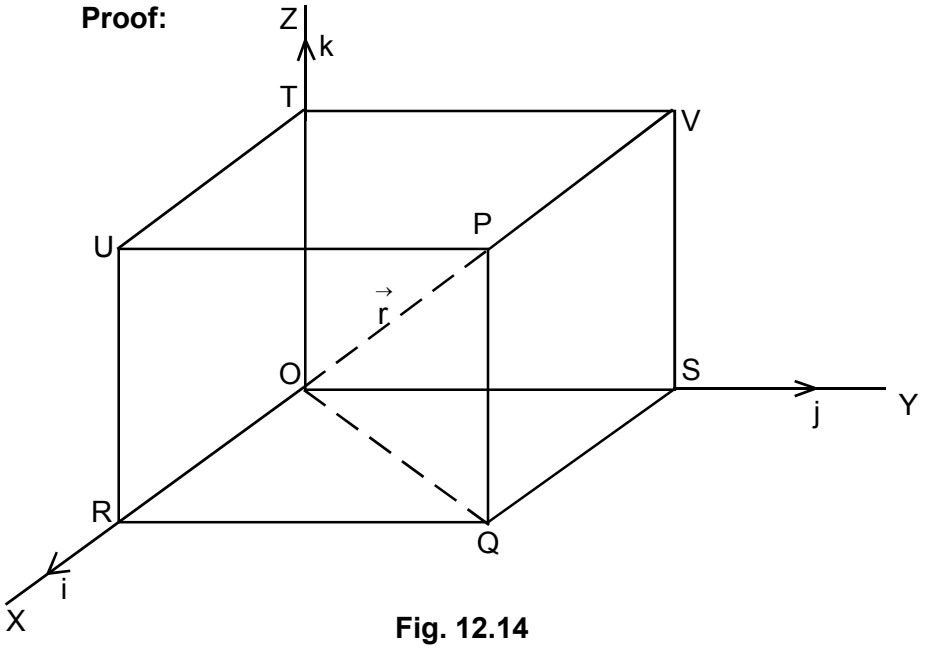


Fig. 12.14

Let OX, OY, OZ be three mutually perpendicular axes. $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors along OX, OY, OZ respectively. Let P be any point (x, y, z) in space and let $\vec{OP} = \vec{r}$.

Draw PQ perpendicular to XOY plane and QR perpendicular to OX.

Then, OR = x; RQ = y; QP = z.

$\therefore \vec{OR} = x\hat{i}$, $\vec{RQ} = y\hat{j}$, $\vec{QP} = z\hat{k}$

Now, $\vec{OP} = \vec{OQ} + \vec{QP} = \vec{OR} + \vec{RQ} + \vec{QP}$

Thus if P is a point (x, y, z) and $\vec{r}$ is the position vector of P, then,

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

From the right angled triangle OQP, $OP^2 = OQ^2 + QP^2$

From the right angled triangle ORQ, $OQ^2 = OR^2 + RQ^2$

$\therefore OP^2 = OR^2 + RQ^2 + QP^2$

$\Rightarrow OP^2 = x^2 + y^2 + z^2$

$\Rightarrow OP = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow r = \sqrt{x^2 + y^2 + z^2}$

$\therefore r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Addition, Subtraction and Multiplication of a vector by a scalar and equality in terms of components:

Let, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be any two vectors, then,

i) $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

ii) $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

iii) $k\vec{a} = k(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = ka_1\hat{i} + ka_2\hat{j} + ka_3\hat{k}$, where, k is scalar.

iv) $\vec{a} = \vec{b} \Leftrightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3$.

v) Two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists a non zero scalar λ such that, $\vec{b} = \lambda\vec{a}$.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then the two vectors are collinear if and only if,

$b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$\Leftrightarrow b_1 = \lambda a_1, b_2 = \lambda a_2, b_3 = \lambda a_3$

$\Leftrightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$

Vector joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2):

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points.

Then, $\vec{AB} = \vec{OB} - \vec{OA}$

$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Magnitude of $\vec{AB} = |\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

ILLUSTRATIVE EXAMPLES:

Example 1: Let O be the origin and let P(4, –3) be a point in the xy-plane. Express $\vec{OP}$ in terms of vectors $\hat{i}$ and $\hat{j}$. Also, find $|\vec{OP}|$.

Solution: The position vector of P is: $-4\hat{i} + 3\hat{j}$.

$\therefore \vec{OP} = -4\hat{i} + 3\hat{j}$

$\Rightarrow |\vec{OP}| = \sqrt{(-4)^2 + 3^2} = 5$

Example 2: Find the vector $\vec{PQ}$ where, P(2, 3) and Q(5, –3). Also find the magnitude of $\vec{PQ}$.

Solution: $\vec{PQ}$ = P.V. of Q – P.V. of P

$= (5\hat{i} - 3\hat{j}) - (2\hat{i} + 3\hat{j})$

$= 3\hat{i} - 6\hat{j}$.

Also, $|\vec{PQ}| = |3\hat{i} - 6\hat{j}| = \sqrt{3^2 + (-6)^2} = \sqrt{45}$

Example 3: If $\vec{a} = 2\hat{i} - 3\hat{j}$, $\vec{b} = 3\hat{i} - 2\hat{j}$, and $\vec{c} = \hat{i} + \hat{j}$; find the components of the vector $\vec{a} - 2\vec{b} + \vec{c}$.

Solution: Here, $\vec{a} - 2\vec{b} + \vec{c} = (2\hat{i} - 3\hat{j}) - 2(3\hat{i} + 2\hat{j}) + (\hat{i} + \hat{j})$

$= 3\hat{i} - 6\hat{j}$

$\therefore$ Components of $\vec{a} - 2\vec{b} + \vec{c}$ are –3, –6.

Example 4: If $\vec{a} = \hat{i} - 2\hat{j}$ and $\vec{b} = 2\hat{i} - 2\hat{j}$; find a unit vector parallel to the vector $3\vec{a} - 2\vec{b}$.

Solution: Let, $\vec{v} = 3\vec{a} - 2\vec{b} = 3(\hat{i} - 2\hat{j}) - 2(2\hat{i} - 2\hat{j})$

$= 3\hat{i} - 6\hat{j} - 4\hat{i} - 4\hat{j}$

$= -\hat{i} - 10\hat{j}$

$\therefore |\vec{v}| = \sqrt{(-1)^2 + (-10)^2} = \sqrt{101}$

Now, $\vec{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{-\hat{i}-10\hat{j}}{\sqrt{101}}$

$\therefore$ A unit vector parallel to $\vec{v}$ is $\pm\vec{v}$ i.e., $\pm\frac{-\hat{i}-10\hat{j}}{\sqrt{101}}$.

Example 5: If $|\vec{a}| = 2$, find $|4\vec{a}|$, $|0\vec{a}|$ and $|(-5)\vec{a}|$.

Solution: Given $|\vec{a}| = 2$

$\therefore |4\vec{a}| = 4|\vec{a}| = 4 \times 2 = 8$,

$|0\vec{a}| = |\vec{0}| = 0$ and $|(-5)\vec{a}| = |(-5)||\vec{a}| = 5 \times 2 = 10$.

Example 6: Find the position of a point which divides the join of two points, whose position vectors are given by $\vec{x}$ and $\vec{y}$ in the ratio 2 : 3 internally.

Solution: Let $\vec{r}$ be the position vector of the point.

$\therefore \vec{r} = \frac{3\vec{x}+2\vec{y}}{3+2}$

$= \frac{1}{5}\left(3\vec{x}+2\vec{y}\right)$

Example 7: Find a vector of magnitude 5 units which is parallel to the vector $2\hat{i}-\hat{j}$.

Solution: Let, $\vec{a} = 2\hat{i}-\hat{j}$, Then, $|\vec{a}| = \sqrt{2^2+(-1)^2} = \sqrt{5}$

$\therefore$ Unit vector parallel to $\vec{a} = \hat{a} = \frac{1}{|\vec{a}|}\vec{a} = \frac{1}{\sqrt{5}}(2\hat{i}-\hat{j}) = \frac{2}{\sqrt{5}}\hat{i}-\frac{1}{\sqrt{5}}\hat{j}$

So, required vector $= 5\hat{a} = 5\left(\frac{2}{\sqrt{5}}\hat{i}-\frac{1}{\sqrt{5}}\hat{j}\right) = 2\sqrt{5}\hat{i}-\sqrt{5}\hat{j}$.

Example 8: Let $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b} = -\hat{i}+\hat{j}$. Find a vector in the direction of $\vec{a}+\vec{b}$ that has magnitude 7 units.

Solution: Here, $\vec{a}+\vec{b} = \hat{i}+2\hat{j}+3\hat{k}+(-\hat{i}+\hat{j}) = 3\hat{j}+3\hat{k} = \vec{c}$

The unit vector in the diretion of $\vec{c} = \vec{a}+\vec{b}$ is:

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{18}}(3\hat{j} + 3\hat{k})$$

$$= \frac{3}{3\sqrt{2}}\hat{j} + \frac{3}{3\sqrt{2}}\hat{k}$$

$$= \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$

Therefore, the vector having magnitude equal to 7 and in the direction of $\hat{c}$ is: $7\left(\frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) = \frac{7}{\sqrt{2}}\hat{j} + \frac{7}{\sqrt{2}}\hat{k}$.

Example 9: If $\vec{a}$ and $\vec{b}$ are position vectors of the points (1, –1) and (–2, m) respectively, then find the value of *m* for which $\vec{a}$ and $\vec{b}$ are collinear.

Solution: Let, $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = -2\hat{i} + m\hat{j}$

Two vectors $\vec{a}$ and $\vec{b}$ are collinear if there exists a non zero scalar λ, such that, $\vec{b} = \lambda\vec{a}$ i.e., $\hat{i} - \hat{j} = \lambda(-2\hat{i} + m\hat{j}) = -2\lambda\hat{i} + m\lambda\hat{j}$

Comparing the component on both sides, we get,

$1 = -2\lambda$, $-1 = m\lambda$

$\Rightarrow \lambda = -\frac{1}{2}$, $m = -\frac{1}{\lambda}$

$\Rightarrow \lambda = -\frac{1}{2}$, $m = 2$

Example 10: Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $2\hat{i} + \hat{j} - \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ in the ratio 2 : 1 (i) internally (ii) externally.

Solution: i) Position vector of R which divides PQ in the ratio 2 : 1 internally is:

$$\vec{OR} = \frac{2\left(\hat{i} + 2\hat{j} + \hat{k}\right) + 1\left(2\hat{i} + \hat{j} - \hat{k}\right)}{2+1}$$

$$= \frac{4\hat{i} + 5\hat{j} + \hat{k}}{3} = \frac{4}{3}\hat{i} + \frac{5}{3}\hat{j} + \frac{1}{3}\hat{k}$$

ii) Position vector of R which divides PQ in the ratio 2 : 1 externally is:

$$\vec{OR} = \frac{2\left(\hat{i}+2\hat{j}+\hat{k}\right)-1\left(2\hat{i}+\hat{j}-\hat{k}\right)}{2-1}$$

$$= 3\hat{j} + 3\hat{k}$$



CHECK YOUR PROGRESS

Q.3: Given $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{b} = 2\vec{i} - 4\vec{j} - 3\vec{k}$, find the magnitudes of:

(i) $\vec{a}$, (ii) $\vec{b}$, (iii) $\vec{a}+\vec{b}$, (iv) $\vec{a}-\vec{b}$.

Q.4: Find the unit vector in the direction of $3\vec{i} - 6\vec{j} + 2\vec{k}$.

Q.5: Show that vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = 4\hat{i} + 6\hat{j} - 2\hat{k}$ are collinear.



12.8 LET US SUM UP

- A quantity which has magnitude only is called a scalar.
- A quantity which has both magnitude and direction is called a vector.
- A vector whose magnitude is the length PQ and whose direction is from P to Q is denoted by $\vec{PQ}$ and its magnitude is denoted by $|\vec{PQ}|$ = PQ.
- A vector whose magnitude is zero and direction is indeterminate is called the zero or null vector.It is denoted by $\vec{0}$.
- A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. The unit vector in the direction of a given vector $\vec{a}$ is denoted by $\hat{a}$ and defined as, $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.
- A vector whose magnitude is equal to another vector $\vec{a}$ but of opposoite direction is called negative of the given vector and is denoted by $-\vec{a}$.

- Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, written as $\vec{a} = \vec{b}$, if they have the (i) same magnitude (ii) same direction.
- If the vectors are represented by two adjacent sides of a parallelogram, then diagonal of the parallelogram will represent the sum of the vectors.
- Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let P be a point dividing AB internally in the ratio m : n. Then the position vector of P is given by $\vec{OP} = \frac{m\vec{b}+n\vec{a}}{m+n}$
- Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let P be a point dividing AB externally in the ratio m : n.

  Then the position vector of P is given by $\vec{OP} = \frac{m\vec{b}+n\vec{a}}{m+n}$
- If a point P in space has coordinate (x, y, z) then its position vector $\vec{r}$ is $x\hat{i} + y\hat{j} + z\hat{k}$ and $r = \sqrt{x^2+y^2+z^2}$ where, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along OX, OY and OZ repectively.
- Two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists a non zero scalar λ such that, $\vec{b} = \lambda\vec{a}$.



12.9 FURTHER READING

1) Kishan, Hari. *Vector Algebra and Calculus*. New Delhi: Atlantic Publishers & Distributors (P) Ltd.
2) Narayan, Shanti. *A Textbook of Vector Calculus*.
3) Pandey, R. K. *Vector Analysis*. New Delhi: Discovery Publishing House.
4) Sharma, Anil Kumar. *Textbook of Vector Calculus*. New Delhi: Discovery Publishing House.
5) Tallack, J. C. *Introduction to Vector Analysis*. Cambridge University Press.
6) *Vector Algebra*. New Delhi: Laxmi Publications(P) Ltd.

12.10 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: a) Density is a scalar quantity as it has only magnitude.

b) Weight is a vector quantity as it has both magnitude as well as direction.

c) Momentum is a vector quantity as it has both magnitude as well as direction.

d) Work done is a scalar quantity as it has only magnitude.

e) Volume is a scalar quantity as it has only magnitude.

f) Acceleration is a vector quantity as it has both magnitude as well as direction.

Ans. to Q. No. 2:

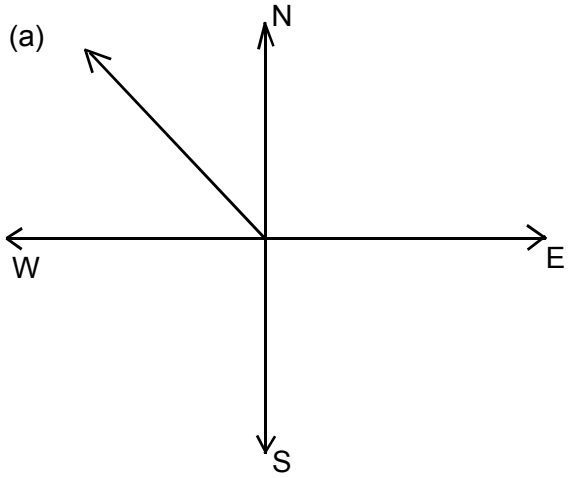


Fig 12.15

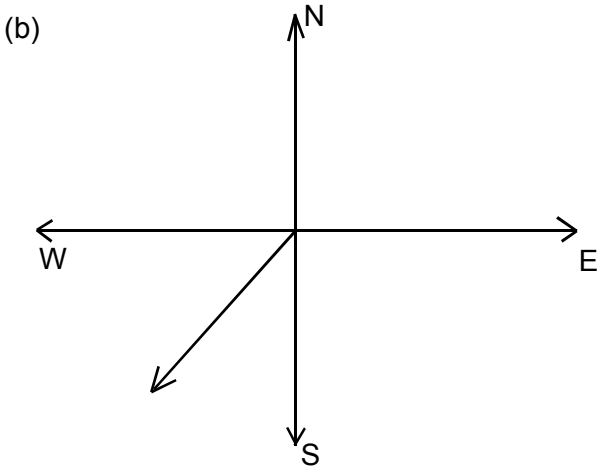


Fig 12.16

Ans. to Q. No. 3: i) $|\vec{a}| = |\vec{i} - \vec{j} + \vec{k}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

ii) $|\vec{b}| = |2\vec{i} - 4\vec{j} - 3\vec{k}| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{38}$

iii) $\vec{a} + \vec{b} = (\hat{i} - \hat{j} + \hat{k}) + (2\hat{i} - 4\hat{j} - 3\hat{k}) = 3\hat{i} - 5\hat{j} - 2\hat{k}$

$\therefore |\vec{a} + \vec{b}| = |3\hat{i} - 5\hat{j} - 2\hat{k}|$

$= \sqrt{3^2 + (-5)^2 + (-2)^2}$

$= \sqrt{38}$

iv) $\vec{a} - \vec{b} = (\hat{i} - \hat{j} + \hat{k}) - (2\hat{i} - 4\hat{j} - 3\hat{k}) = \hat{i} + 3\hat{j} + 4\hat{k}$

$$\therefore |\vec{a} - \vec{b}| = |-\hat{i} + 3\hat{j} + 4\hat{k}|$$
$$= \sqrt{(-1)^2 + 3^2 + 4^2}$$
$$= \sqrt{26}$$

Ans. to Q. No. 4: Let, $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$

Then, $|\vec{a}| = |3\hat{i} - 6\hat{j} + 2\hat{k}|$
$$= \sqrt{3^2 + (-6)^2 + 2^2}$$
$$= 7$$

$\therefore$ Unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{7}$$

$$= \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$$

Ans. to Q. No. 5: Here $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = 4\hat{i} + 6\hat{j} - 2\hat{k}$

Now, $\vec{b} = 2(2\hat{i} + 3\hat{j} - \hat{k}) = 2\vec{a}$

$\therefore$ $\vec{b}$ is a scalar multiple of $\vec{a}$.

Hence, $\vec{a}$ and $\vec{b}$ are collinear vectors.



12.11 MODEL QUESTIONS

Q.1: Find a unit vector parallel to $\hat{i} - \hat{j} + \hat{k}$.

Q.2: If the position vectors of A and B are $2\hat{i} + 3\hat{j} - 7\hat{k}$ and $4\hat{i} - 3\hat{j} + 4\hat{k}$ repectively, find $\vec{AB}$ and $|\vec{AB}|$. Also determine direction cosines of $\vec{AB}$.

Q.3: Prove that the vectors $2\hat{i} + 3\hat{j} - 6\hat{k}$, $6\hat{i} - 2\hat{j} + 2\hat{k}$ and $3\hat{i} + 6\hat{j} - 2\hat{k}$ form the sides of an equilateral triangle.

Q.4: Find the magnitude of each of the following vectors :

i) $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

ii) $\vec{b} = \hat{i} + \hat{j} + \hat{k}$

iii) $\vec{c} = \frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

Q.5: Find the unit vector in the direction of vector $\vec{PQ}$ where, P and Q are the points (1, 2, 3) and (4, 5, 6) respectively.

Q.6: Find the position vectors of the points which divide the join of the points $2\vec{a} - 3\vec{b}$ and $3\vec{a} - 2\vec{b}$ internally and externally in the ratio 2 : 3.

Q.7: Find the position vector of the mid-point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2).

Q.8: Find the position vector of a point R which divides the line joining points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} - \hat{k}$ respectively in the ratio 2 : 1 (i) internally (ii) externally.

Q.9: Find the direction cosines of the vector joining the points A(1, 2, –3) and B(–1, –2, 1) directed from A to B.

Q.10: Let A be a point (–3, 2) and O the origin of xy plane.Express $\vec{OA}$ in terms of $\hat{i}$ and $\hat{j}$ and find $|\vec{OA}|$.

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UNIT 13: PRODUCT OF VECTORS–I

UNIT STUCTURE

13.1 Learning Objectives

13.2 Introduction

13.3 Product of Two Vectors

13.3.1 Scalar or Dot Product

13.3.2 Vector (Cross) Product

13.6 Let Us Sum Up

13.7 Further Reading

13.8 Answers to Check Your Progress

13.9 Model Questions

13.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define scalar (or dot) and vector (or cross) product of two vectors
- find angle between two vectors
- find projection of a vector on another vector
- use cross product to find area of a triangle and parallelogram.

13.2 INTRODUCTION

In unit 12, we learnt how to add and subtract two vectors, and how to multiply a vector by a scalar. In this unit, we will study the notion of another operation ‘product’. We will discuss scalar and vector product of two vectors.

13.3 PRODUCT OF TWO VECTORS

The product of two vectors is defined in two ways, viz, scalar product (or dot product) and vector (or cross) product. The result of a scalar product of two vectors is a scalar and vector product of two vectors is a vector.

13.3.1 Scalar or Dot Product

Let $\vec{a}$ and $\vec{b}$ be two non zero vectors and θ be is the angle between $\vec{a}$ and $\vec{b}$. Then the scalar product $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ and is defined by

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta,\ 0 \le \theta \le \pi$

Geometrical Interpretation of scalar product:

Let, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. Let θ be the angle between $\vec{a}$ and $\vec{b}$. We draw BL perpendicular to OA.OL is called the projection of $\vec{b}$ on $\vec{a}$.

From ΔOLB, $\cos\theta = \dfrac{OL}{OB}$

$\Rightarrow$ OL = (OB)$\cos\theta$

$\Rightarrow$ OL = $|\vec{b}|\cos\theta$ ... ...(1)

By definition, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{a}|$OL [using (1)]

$\therefore$ OL = $\dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$

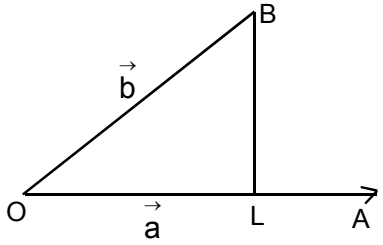


Fig 13.1

Projection of $\vec{b}$ on $\vec{a}$ = $\dfrac{\vec{a}.\vec{b}}{|\vec{a}|} = \dfrac{\vec{a}}{|\vec{a}|}.\vec{b} = \hat{a}.\vec{b}$

Also,projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a}.\vec{b}}{|\vec{b}} = \vec{a}.\dfrac{\vec{b}}{|\vec{b}} = \vec{a}.\hat{b}$

Important Observations:

1) Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors, then, $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$.
2) Dot product of two equal vectors,

   $\vec{a}.\vec{a} = |\vec{a}||\vec{a}|\cos\theta = |\vec{a}||\vec{a}| = |\vec{a}|^2 = \vec{a}$

   Also, $(\vec{a})^2 = \vec{a}.\vec{a} = |\vec{a}|^2 = a^2$.

3) The dot product $\vec{a} . \vec{b}$ may be positive, negative or zero.

   i) If the angle between the two vectors is acute (i.e., $0 < \theta < 90^0$), then $\cos\theta$ is positive.In this case, dot product is positive.

   ii) If the angle between the two vectors is obtuse (i.e., $90^0 < \theta < 180^0$), then $\cos\theta$ is negative.In this case, dot product is negative.

   iii) If the angle between the two vectors is 90^0 (i.e., $\theta = 90^0$) then, $\cos\theta = \cos 90^0 = 0$. In this case,dot product is zero.

4) Scalar product of collinear vectors:

   i) when the vectors $\vec{a}$ and $\vec{b}$ are collinear and in the same direction, then, $\theta = 0^0$.

   Thus, $\vec{a} . \vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|(1) = |\vec{a}||\vec{b}| = ab$.

   ii) when the vectors $\vec{a}$ and $\vec{b}$ are collinear and in the opposite direction, then, $\theta = \pi$.

   Thus, $\vec{a} . \vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\pi = -|\vec{a}||\vec{b}| = -ab$.

Properties of Scalar Product of Two Vectors:

1) The scalar product of two vectors is commutative

   i.e., $\vec{a} . \vec{b} = \vec{b} . \vec{a}$ for any two vectors $\vec{a}$ and $\vec{b}$.

2) The scalar product is distributive over addition.

   i.e., $\vec{a} . (\vec{b} + \vec{c}) = \vec{a} . \vec{b} + \vec{a} . \vec{c}$ for any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

3) If 'm' is any scalar and $\vec{a}$, $\vec{b}$ are two vectors, then,

   $(m.\vec{a}).\vec{b} = m(\vec{a} . \vec{b}) = \vec{a} . (m\vec{b})$

4) If m, n are scalars and $\vec{a}$, $\vec{b}$ are two vectors, then,

   $m\vec{a} . n\vec{b} = mn(\vec{a} . \vec{b}) = (mn\vec{a}).\vec{b} = \vec{a} . (mn\vec{b})$

Note: 1) $\hat{i} . \hat{i} = \hat{j} . \hat{j} = \hat{k} . \hat{k} = 1$

We have, $\hat{i} . \hat{i} = |\hat{i}||\hat{i}|\cos 0 = (1)(1)(1) = 1$

Similarly, $\hat{j} . \hat{j} = \hat{k} . \hat{k} = 1$.

2) $\hat{i} . \hat{j} = \hat{j} . \hat{k} = \hat{k} . \hat{i} = 0$

We have, $\hat{i}.\hat{j} = |\hat{i}||\hat{j}|\cos 90 = (1)(1)(0) = 0$.

Similarly, $\hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$.

3) For any two vectors $\vec{a}$ and $\vec{b}$

i) $(\vec{a}+\vec{b})^2 = (\vec{a})^2 + 2\vec{a}.\vec{b} + (\vec{b})^2 = a^2 + 2\vec{a}.\vec{b} + b^2$

ii) $(\vec{a}-\vec{b})^2 = (\vec{a})^2 + 2\vec{a}.\vec{b} + (\vec{b})^2 = a^2 - 2\vec{a}.\vec{b} + b^2$

iii) $(\vec{a}+\vec{b})(\vec{a}-\vec{b}) = (\vec{a})^2 - (\vec{b})^2 = a^2 - b^2$

4) If $\vec{a}.\vec{b} = 0$, we have three possibilities

i) $|\vec{a}| = 0$ i.e., $\vec{a}$ is a zero vector and $\vec{b}$ any vector.

ii) $|\vec{b}| = 0$ i.e., $\vec{b}$ is a zero vector and $\vec{a}$ any vector.

iii) $\cos\theta = 0 \Rightarrow \theta = 90^0 \Rightarrow \vec{a} \perp \vec{b}$, $\vec{a} \neq 0$, $\vec{b} \neq 0$

Scalar Product in Terms of Components:

Let, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then their scalar product is given as:

$$\vec{a}.\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1\hat{i}.(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{j}.(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k}.(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1b_1(\hat{i}.\hat{i}) + a_1b_2(\hat{i}.\hat{j}) + a_1b_3(\hat{i}.\hat{k}) + a_2b_1(j.\hat{i}) + a_2b_2(\hat{j}.\hat{j}) + a_2b_2(\hat{j}.\hat{k}) + a_3b_1(\hat{k}.\hat{i}) + a_3b_2(\hat{k}.\hat{j}) + a_3b_3(\hat{k}.\hat{k})$$

$$= a_1b_1 + a_2b_2 + a_3b_3$$

Thus, $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

The scalar product of two vectors is equal to the sum of the products of their corresponding components.

Angle Between Two Vectors: Let $\vec{a}$, $\vec{b}$ be two vectors inclined at angle θ.

Then, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \theta = \cos^{-1}\left[\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right]$$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then, $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$; $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

$$\therefore \theta = \cos^{-1}\left[\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right].$$

ILLUSTRATIVE EXAMPLES:

Examples 1: Find $\vec{a}.\vec{b}$ when,

i) $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$

ii) $\vec{a} = \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{k}$

iii) $\vec{a} = \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 2\hat{k}$

Solution: i) $\vec{a}.\vec{b} = (\hat{i} - 2\hat{j} + \hat{k}).(4\hat{i} - 4\hat{j} + 7\hat{k})$

$= (1).(4) + (-2).(-4) + (1).(7)$

$= 19$

ii) $\vec{a}.\vec{b} = (\hat{j} + 2\hat{k}).(2\hat{i} + \hat{k})$

$= (0).(2) + (1).(0) + (2).(1)$

$= 2$

iii) $\vec{a}.\vec{b} = (\hat{j} - 2\hat{k}).(2\hat{i} + 3\hat{j} - 2\hat{k})$

$= (0).(2) + (1).(3) + (-2).(-2)$

$= 7$

Example 2: Find angle 'θ' between the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

Solution: The angle 'θ' between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Now, $\vec{a}.\vec{b} = (\hat{i} + \hat{j} - \hat{k}).(\hat{i} - \hat{j} + \hat{k}) = 1 - 1 - 1 = -1$

$|\vec{a}| = \sqrt{1^2+1^2+(-1)^2} = \sqrt{3}$, $|\vec{b}| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$

Therefore, we have $\cos\theta = -\frac{1}{3}$

$\therefore \theta = \cos^{-1}\left(-\frac{1}{3}\right)$.

Example 3: For what values of 'm' the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other?

i) $\vec{a} = m\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 4\hat{i} - 9\hat{j} + 2\hat{k}$

ii) $\vec{a} = 5\hat{i} - 9\hat{j} + 2\hat{k}$ and $\vec{b} = m\hat{i} + 2\hat{j} + \hat{k}$

Solution: i) Given $\vec{a}.\vec{b} = 0$

$\Rightarrow (m\hat{i} + 2\hat{j} + \hat{k}).(4\hat{i} - 9\hat{j} + 2\hat{k}) = 0$

$\Rightarrow 4m - 18 + 2 = 0$

$\Rightarrow 4m - 16 = 0$

$\Rightarrow 4m = 16$

$\Rightarrow m = 4$

ii) Given $\vec{a}.\vec{b} = 0$

$\Rightarrow 5m - 18 + 2 = 0$

$\Rightarrow 5m - 16 = 0$

$\Rightarrow m = \frac{16}{5}$

Example 4: Find the projection of the vector $7\hat{i} + \hat{j} - 4\hat{k}$ on $2\hat{i} + 6\hat{j} + 3\hat{k}$.

Solution: Let, $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

$\therefore$ Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{\left(7\hat{i}+\hat{j}-4\hat{k}\right).\left(2\hat{i}+6\hat{j}+3\hat{k}\right)}{\sqrt{2^2+6^2+3^2}}$

$= \frac{14+6-12}{\sqrt{4+36+9}}$

$= \frac{8}{7}$.

Example 5: If $\vec{a}$ and $\vec{b}$ are unit vectors inclined at an angle θ, then, prove that $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|$.

Solution: $|\hat{a}-\hat{b}|^2 = \hat{a}^2 + \hat{b}^2 - 2\hat{a}.\hat{b} = 1 + 1 - 2|\hat{a}||\hat{b}|\cos\theta$

$= 2 - 2\cos\theta$

$= 2(1 - 2\cos\theta)$

$= 2\left(2\sin^2\frac{\theta}{2}\right)$

$\therefore |\hat{a}-\hat{b}| = 2\sin\frac{\theta}{2}$

$\Rightarrow \sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|$

Example 6: If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution: Given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow (\vec{a} + \vec{b})^2 = (-\vec{c})^2$

$\Rightarrow (\vec{a})^2 + (\vec{b})^2 + 2\vec{a}.\vec{b} = (\vec{c})^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta = |\vec{c}|^2$

$\Rightarrow 3^2 + 5^2 + 2(3)(5)\cos\theta = 7^2$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$.

Example 7: Find $|\vec{a}|$ and $|\vec{b}|$ if $(\vec{a} - \vec{b}).(\vec{a} + \vec{b}) = 27$ and $|\vec{a}| = 2|\vec{b}|$.

Solution: We have, $(\vec{a} - \vec{b}).(\vec{a} + \vec{b}) = 27$

$\Rightarrow 4|\vec{b}|^2 - |\vec{b}|^2 = 27$

$\Rightarrow 3|\vec{b}|^2 = 27$

$\Rightarrow |\vec{b}|^2 = 9$

$\Rightarrow |\vec{b}| = 3$

$\Rightarrow |\vec{a}| = 2|\vec{b}| = 2 \times 3 = 6$

Thus, $|\vec{a}| = 6$ and $|\vec{b}| = 3$.

Example 8: Find the projection of $\vec{a}$ on $\vec{b}$ where, $\vec{a} = 3\hat{i} - 5\hat{j} + 2\hat{k}$ and $\vec{b} = 7\hat{i} + \hat{j} - 2\hat{k}$.

Solution:

$$\therefore \text{ Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{\left(3\hat{i}-5\hat{j}+2\hat{k}\right).\left(7\hat{i}+\hat{j}-2\hat{k}\right)}{\sqrt{7^2+1^2+(-2)^2}}$$

$$= \frac{3.7+(-5)1+2.(-2)}{\sqrt{54}}$$

$$= \frac{12}{\sqrt{54}}$$



CHECK YOUR PROGRESS

Q.1: Find the angle between the vectors:

i) $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = -4\hat{i} + 2\hat{k}$

ii) $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = -2\hat{i} + 2\hat{j} + 4\hat{k}$

Q.2: If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3, |\vec{b}| = 5, |\vec{c}| = 7$. Find the angle between $\vec{a}$ and $\vec{b}$.

13.3.2 Cross or Vector Product

The vector product of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ (read as $\vec{a}$ cross $\vec{b}$) and is defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$, where, θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

Properties of the Vector Product:

1) Vector product is not commutative:

If $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right-handed system,then $-\hat{n}$ is a unit vector perpendicular to the plane of $\vec{b}$ and $\vec{a}$ such that $\vec{b}$, $\vec{a}$, $-\hat{n}$ form a right handed system.

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

$\vec{b} \times \vec{a} = |\vec{b}||\vec{a}|\sin\theta(-\hat{n}) = -|\vec{b}||\vec{a}|\sin\theta\,\hat{n} = -\vec{a} \times \vec{b}$

$\therefore \vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$.

2) Vector product is distributive with repect to vector addition i.e.,

$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

and $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$

3) Vector product of two parallel vectors is zero vector.

If two vectors $\vec{a}$ and $\vec{b}$ are parallel, then θ, the angle between them is 0 or π so that $\sin\theta = 0$.

$\therefore \vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n} = 0$

Conversely, if the vector product of two non zero vectors is zero vector, then the two vectors are parallel.

4) $\hat{i} \times \hat{i} = |\hat{i}||\hat{i}|\sin\theta\,\hat{k} = \vec{0}$

Similarly, $\hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

In particular, $\vec{a} \times \vec{a} = \vec{0}$

5) We have, $\hat{i} \times \hat{j} = |\hat{i}||\hat{j}|\sin 90^0 \hat{k} = \hat{k}$, Since $\hat{i}$, $\hat{j}$, $\hat{k}$ form a right handed system.

$\therefore \hat{j} \times \hat{i} = -\hat{k}$

Similarly, $\hat{j} \times \hat{k} = \hat{i}$ $\quad\therefore \hat{k} \times \hat{j} = -\hat{i}$

and $\hat{k} \times \hat{i} = \hat{j}$ $\quad\therefore \hat{i} \times \hat{k} = -\hat{j}$

6) Let, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then their vector product is given as:

$$\vec{a} \times \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$= a_1 b_1 \hat{i} \times \hat{i} + a_1 b_2 \hat{i} \times \hat{j} + a_1 b_3 \hat{i} \times \hat{k} + a_2 b_1 \hat{j} \times \hat{i} + a_2 b_2 \hat{j} \times \hat{j} +$$

$$a_2 b_3 \hat{j} \times \hat{k} + a_3 b_1 \hat{k} \times \hat{i} + a_3 b_2 \hat{k} \times \hat{j} + a_3 b_3 \hat{k} \times \hat{k}$$

$$= 0 + a_1 b_2 \hat{k} - a_1 b_3 \hat{j} - a_2 b_1 \hat{k} + 0 + a_2 b_3 \hat{i} + a_3 b_1 \hat{j} - a_3 b_2 \hat{i} + 0$$

$$= (a_2 b_3 - a_3 b_2) \hat{i} + (a_3 b_1 - a_1 b_3) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

7) Relation between scalar and vector product

$$(\vec{a} \times \vec{b})^2 = |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

$$= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta)$$

$$= |\vec{a}|^2 |\vec{b}|^2 - (|\vec{a}||\vec{b}|\cos^2\theta)^2$$

$$= a^2 b^2 - (\vec{a} . \vec{b})^2$$

$$\therefore (\vec{a} \times \vec{b})^2 = a^2 b^2 - (\vec{a} . \vec{b})^2$$

8) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is given as $\frac{1}{2}|\vec{a} \times \vec{b}|$.

Let $\vec{AC} = \vec{a}$ and $\vec{AB} = \vec{b}$

Also, CD = $|\vec{a}|\sin\theta$

Area of triangle ABC = $\frac{1}{2}$AB.CD

$$= \frac{1}{2}|\vec{b}||\vec{a}|\sin\theta$$

$$= \frac{1}{2}|\vec{a} \times \vec{b}|$$

9) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a} \times \vec{b}|$.

Let, $\vec{AD} = \vec{a}$ and $\vec{AB} = \vec{b}$

Also DE = $|\vec{a}|\sin\theta$

Area of parallelogram ABCD = AB.DE.

$$= |\vec{b}||\vec{a}|\sin\theta$$

$$= |\vec{a} \times \vec{b}|$$

ILLUSTRATIVE EXAMPLES:

Example 1: Find $\vec{a} \times \vec{b}$ if $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Solution: We have, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$$= (3 - 2)\hat{i} - (3 - 1)\hat{j} + (2 - 1)\hat{k}$$

$$= \hat{i} - 2\hat{j} + \hat{k}$$

Example 2: Find a unit vector perpendicular to both the vectors $\vec{a} = 4\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$.

Solution: A vector perpendicular to both $\vec{a}$ and $\vec{b}$ is:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = (-1 - 3)\hat{i} - (-8 + 6)\hat{j} + (4-2)\hat{k}$$

$$= -5\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{(-5)^2 + 1^2 + 6^2} = \sqrt{62}$$

$$\therefore \text{ required unit vector} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{1}{62}\left(-5\hat{i} + 2\hat{j} + 6\hat{k}\right).$$

Example 3: Show that $|\vec{a} \times \vec{b}|^2 = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2$

Solution: We know that $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

$$\therefore |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$\Rightarrow |\vec{a} \times \vec{b}|^2 = (|\vec{a}||\vec{b}|\sin\theta)^2$$

$$\Rightarrow |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta)$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (|\vec{a}|^2 |\vec{b}|^2 \cos^2\theta)$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (|\vec{a}||\vec{b}|\cos\theta)^2$

$\therefore |\vec{a} \times \vec{b}|^2 = (\vec{a}.\vec{a})(\vec{b}.\vec{b}) - (\vec{a}.\vec{b})^2$

Example 4: Find $|\vec{a} \times \vec{b}|$ if $|\vec{a}| = 10$, $|\vec{b}| = 2$, $\vec{a}.\vec{b} = 12$.

Solution: Given $|\vec{a}| = 10$, $|\vec{b}| = 2$, $\vec{a}.\vec{b} = 12$

$$\therefore \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{12}{10.2} = \frac{3}{4}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

Now, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta = 10 \times 2 \times \frac{4}{5} = 16$



13.6 LET US SUM UP

- Let $\vec{a}$ and $\vec{b}$ be two non zero vectors and θ be the angle between $\vec{a}$ and $\vec{b}$. Then the scalar product $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ and is defined by $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, $0 \le \theta \le \pi$.
- Projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|} = \frac{\vec{a}}{|\vec{a}|}.\vec{b} = \hat{a}.\vec{b}$
- Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \vec{a}\frac{\vec{b}}{|\vec{b}|} = \vec{a}.\hat{b}$
- Let, $\vec{a}$ and $\vec{b}$ be two non-zero vectors, then, $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$.
- Let, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then,

  $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

  i.e., The scalar product of two vectors is equal to the sum of the products of their corresponding components.

- Let $\vec{a}$, $\vec{b}$ be two vectors inclined at angle θ, then,

$$\theta = \cos^{-1}\left[\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right]$$

- The vector product of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where, θ is the angle between $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

- Let, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$



13.7 FURTHER READINGS

1) Kishan, Hari. *Vector Algebra and Calculus*. New Delhi: Atlantic Publishers & Distributors (P) Ltd.
2) Narayan, Shanti. *A Textbook of Vector Calculus*.
3) Pandey, R. K. *Vector Analysis*. New Delhi: Discovery Publishing House.
4) Sharma, Anil Kumar. *Textbook of Vector Calculus*. New Delhi: Discovery Publishing House.
5) Tallack, J. C. *Introduction to Vector Analysis*. Cambridge University Press.
6) *Vector Algebra*. New Delhi: Laxmi Publications(P) Ltd.



13.8 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: i) Here $\vec{a}.\vec{b} = 3.(-4) + (-1).0 + 2.2 = -8$

$$|\vec{a}| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{(-4)^2 + 2^2} = \sqrt{20}$$

$$\therefore \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-8}{\sqrt{20}\sqrt{14}} = \frac{-4}{\sqrt{70}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-4}{\sqrt{70}}\right).$$

ii) Try yourself.

Ans. to Q. No. 2: Let θ be the angle between $\vec{a}$ and $\vec{b}$. then,

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a}.\vec{b}}{3\times 5} = \frac{1}{15}\left(\vec{a}.\vec{b}\right) \qquad \text{... ... ...(1)}$$

Since, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow (\vec{a} + \vec{b})^2 = \vec{c}^2$

$\Rightarrow \vec{a} + 2\vec{a}.\vec{b} + \vec{b}^2 = \vec{c}^2$

$\Rightarrow 3^2 + 2\vec{a}.\vec{b} + 5^2 = 7^2$

$\Rightarrow \vec{a}.\vec{b} = \frac{49-34}{2} = \frac{15}{2}$

From (1), $\cos\theta = \frac{1}{15}\left(\frac{15}{2}\right) = \frac{1}{2} \Rightarrow \theta = 60^0$



13.9 MODEL QUESTIONS

Q.1: Show that $|\vec{a}\times\vec{b}|^2 + |\vec{a}.\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$.

Q.2: If $\vec{a} = 2\hat{i} - 3\hat{j} - \hat{k}$, $\vec{b} = \hat{i} + 4\hat{j} - 2\hat{k}$, find $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$.

Q.3: Find the area of the parallelogram having

i) the adjacent sides as $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

ii) the diagonals as $3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$.

Q.4: Prove that: i) $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$

ii) $(\vec{a}.\vec{b})^2 \le |\vec{a}|^2|\vec{b}|^2$.

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UNIT 14: PRODUCT OF VECTORS–II

UNIT STUCTURE

14.1 Learning Objectives

14.2 Introduction

14.4 Product of Three Vectors

- 14.4.1 Scalar Triple Product
- 14.4.2 Vector Triple Product

14.5 Product of Four Vectors

- 14.5.1 Scalar Product of Four Vectors
- 14.5.2 Vector Product of Four Vectors

14.6 Let Us Sum Up

14.7 Further Reading

14.8 Answers to check Your Progress

14.9 Model Questions

14.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define product of three vectors
- know scalar and vector triple product
- define product of four vectors.

14.2 INTRODUCTION

In unit 13, We will discuss scalar and vector product of two vectors. In this unit, we shall discuss the scalar triple product and the vector triple product also. Finally, we will discuss about the product of four vectors.

14.4 PRODUCT OF THREE VECTORS

The products of the type $\vec{a}.(\vec{b} \times \vec{c})$ and $(\vec{a} \times \vec{b}).\vec{c}$ are called triple products of vectors.

14.4.1 Scalar Triple Product

Definition: Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors. Then the product $\vec{a}.(\vec{b} \times \vec{c})$ is called the scalar triple product of $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$.

$\therefore [\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}.(\vec{b} \times \vec{c})$.

Note: $[\vec{a}\ \vec{b}\ \vec{c}]$ is a scalar quantity.

Geometrical interpretation of scalar triple product : The scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ represents numerically the volume of a rectangular parallelopiped having $\vec{a}$, $\vec{b}$, $\vec{c}$ as coterminous edges.

Scalar Triple Product in Terms of Components:

Let, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, and

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$,

then, $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= (b_2c_3 - b_3c_2)\hat{i} - (b_1c_3 - b_3c_1)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}$$

$\therefore \vec{a}.(\vec{b} \times \vec{c}) = (b_2c_3 - b_3c_2)a_1 - (b_1c_3 - b_3c_1)a_2 + (b_1c_2 - b_2c_1)a_3$

$$= \begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of Scalar Triple Product:

1) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors, then,

   $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

2) The value of the scalar triple product remains same in magnitude but changes in sign, if cyclic order of $\vec{a}$, $\vec{b}$ and $\vec{c}$ is changed.

   i.e., $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}]$

3) The scalar triple product is zero if any two of the given vectors are equal.
i.e., $[\vec{a}\ \vec{a}\ \vec{c}] = (\vec{a} \times \vec{a}).\vec{c} = 0$

4) For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and scalar λ, we have,
$[\lambda\vec{a}\ \vec{b}\ \vec{c}] = \lambda[\vec{a}\ \vec{b}\ \vec{c}]$

5) The scalar triple product is zero if any two of them is parallel or collinear.
i.e., $[\vec{a}\ \vec{b}\ \vec{c}] = [\lambda\vec{b}\ \vec{b}\ \vec{c}]$ if $\vec{a}$ is parallel to $\vec{b}$.

$= \lambda[\vec{b}\ \vec{b}\ \vec{c}]$

$= \lambda.0 = 0$

Example 1: If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$. Find $\vec{a}.(\vec{b} \times \vec{c})$.

Solution: We have, $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 3 & 1 & 1 \end{vmatrix}$

$= \hat{i}(1 - 2) - \hat{j}(-1 - 6) + \hat{k}(-1 - 3)$

$= -\hat{i} + 7\hat{j} - 4\hat{k}$

Now, $\vec{a}.(\vec{b} \times \vec{c}) = (\hat{i} + \hat{j} + \hat{k}).(-\hat{i} + 7\hat{j} - 4\hat{k})$

$= 1.(-1) + 1.7 + 1.(-4)$

$= -1 + 7 - 4 = 2$

Example 2: Find the volume of a parallelopiped whose edges are represented by $\vec{a} = \hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + \hat{k}$

Solution: We have, $\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{vmatrix}$

$= 1(1 + 1) + 3(2 - 3) + 4(2 + 3)$

$= 2 - 3 + 20 = 19$

$\therefore$ Required Volume of parallelopiped (V) $= [\vec{a}\ \vec{b}\ \vec{c}] = 19$.

Example 3: Prove that $[\vec{a}+\vec{b}\ \vec{b}+\vec{c}\ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

Solution: LHS $= [\vec{a}+\vec{b}\ \vec{b}+\vec{c}\ \vec{c}+\vec{a}]$

$= (\vec{a}+\vec{b}).[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$

$= (\vec{a}+\vec{b}).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$

$= (\vec{a}+\vec{b}).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]$ Q $\vec{c}\times\vec{c}=0$

$= \vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})+$

$\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{a})+\vec{b}.(\vec{c}\times\vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{a}]+[\vec{a}\ \vec{c}\ \vec{a}]+[\vec{b}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{b}\ \vec{a}]$

$+[\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{c}]$

$= 2[\vec{a}\ \vec{b}\ \vec{c}]$ = RHS

14.4.2 Vector Triple Product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors. Then, the vectors $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$ are called vector triple products.

Properties of Vector Triple Product:

1) Vector triple product is not associative

   i.e., $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$.

2) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then,

   $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{c}.\vec{a})\vec{c}$

   Similarly, $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

ILLUSTRATIVE EXAMPLES:

Example 1: Prove that

i) $(\vec{a}\times\vec{b})\times\vec{c}+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$

ii) $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k}) = 2\vec{a}$

Solution:

i) L.H.S. $= \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$

$= (\vec{a} . \vec{c})\vec{b} - (\vec{a} . \vec{b})\vec{c} + (\vec{b} . \vec{a})\vec{c} - (\vec{b} . c)\vec{a} +$

$(\vec{c} . \vec{b})\vec{a} - (\vec{c} . \vec{a})\vec{b} = \vec{0}$

ii) Let, $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$\hat{i} \times (\vec{a} \times \hat{i}) = (\hat{i} . \hat{i})\vec{a} - (\hat{i} . \vec{a})\hat{i} = \vec{a} - a_1 \hat{i}$

Similarly, $\hat{j} \times (\vec{a} \times \hat{j}) = \vec{a} - a_2 \hat{j}$ and $\hat{k} \times (\vec{a} \times \hat{k}) = \vec{a} - a_3 \hat{k}$

$\hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) = \hat{k} \times (\vec{a} \times \hat{k})$

$= (\vec{a} - a_1 \hat{i}) + (\vec{a} - a_2 \hat{j}) + (\vec{a} - a_3 \hat{k})$

$= 3\vec{a} - (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})$

$= 3\vec{a} - \vec{a}$

$= 2\vec{a}$

Example 2: Show that:

$\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{d})\} = (\vec{b}, \vec{d})(\vec{a} \times \vec{c}) - (\vec{b} . \vec{c})(\vec{a} \times \vec{d})$

Solution: $\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{d})\} = \vec{a} \times \{(\vec{b} . \vec{d})\vec{c} - (\vec{b} . \vec{c})\vec{d}\}$

$= (\vec{b} . \vec{d})(\vec{a} \times \vec{c}) - (\vec{b} . \vec{c})(\vec{a} \times \vec{d})$

14.4 PRODUCT OF FOUR VECTORS

14.4.1 Scalar Product of Four Vectors

The scalar product $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d})$ is called scalar product of four vectors.

Let, $\vec{c} \times \vec{d} = \vec{e}$, then,

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}).\vec{e}$

$= \vec{a} .(\vec{b} \times \vec{e})$

$= \vec{a} .\{\vec{b} \times (\vec{c} \times \vec{d})\}$

$$= \vec{a}.\{(\vec{b}.\vec{d})\vec{c} - (\vec{b}.\vec{e})\vec{d}\}$$

$$= (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$$

$$= \begin{vmatrix} \vec{a}.\vec{c} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}$$

Example 1: Show that:

$(\vec{b} \times \vec{c}).(\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a}).(\vec{b} \times \vec{d}) + (\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = 0$

Solution:

$(\vec{b} \times \vec{c}).(\vec{a} \times \vec{d}) = (\vec{b}.\vec{a})(\vec{c}.\vec{d}) - (\vec{b}.\vec{d})(\vec{c}.\vec{a})$ ... ...(1)

$(\vec{c} \times \vec{a}).(\vec{b} \times \vec{d}) = (\vec{c}.\vec{b})(\vec{a}.\vec{d}) - (\vec{c}.\vec{d})(\vec{a}.\vec{b})$ ... ...(2)

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c})$ ... ...(3)

Adding (1), (2) and (3), we have,

$(\vec{b} \times \vec{c}).(\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a}).(\vec{b} \times \vec{d}) + (\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = 0$

14.4.2 Vector Product of Four Vectors

The vector product $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is called vector product of four vectors.

Vector product $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ in terms of $\vec{a}$ and $\vec{b}$

Let, $\vec{c} \times \vec{d} = \vec{e}$

then, $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{e}$

$$= (\vec{a}.\vec{e})\vec{b} - (\vec{b}.\vec{e})\vec{a}$$

$$= (\vec{a}.\vec{c} \times \vec{d})\vec{b} - (\vec{b}.\vec{c} \times \vec{d})\vec{a}$$

$$= [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

Vector Product $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ in terms of $\vec{c}$ and $\vec{d}$

Let, $\vec{a} \times \vec{b} = \vec{e}$,

then, $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{e} \times (\vec{c} \times \vec{d})$

$$= (\vec{e}.\vec{d})\vec{e} - (\vec{e}.\vec{c})\vec{d}$$

$$= (\vec{a} \times \vec{b} . \vec{d})\vec{c} - (\vec{a} \times \vec{b} . \vec{c})\vec{d}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

Example: Show that: $[\vec{b} \times \vec{c},\ \vec{c} \times \vec{a},\ \vec{a} \times \vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$

Solution: $[\vec{b} \times \vec{c},\ \vec{c} \times \vec{a},\ \vec{a} \times \vec{b}] = (\vec{b} \times \vec{c}).(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})$

$$= (\vec{b} \times \vec{c}).\{[\vec{c}\ \vec{a}\ \vec{b}]\vec{a} - [\vec{c}\ \vec{a}\ \vec{a}]\vec{b}\}$$

$$= (\vec{b} \times \vec{c}).\{[\vec{c}\ \vec{a}\ \vec{b}]\vec{a} - \vec{0}\}$$

$$= [\vec{a}\ \vec{b}\ \vec{c}](\vec{b} \times \vec{c} . \vec{a})$$

$$= [\vec{a}\ \vec{b}\ \vec{c}][\vec{b}\ \vec{c}\ \vec{a}]$$

$$= [\vec{a}\ \vec{b}\ \vec{c}]^2$$

Example 1: If $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} + \hat{k}$, $\vec{c} = -\hat{i} + \hat{j} + \hat{k}$ and $\vec{d} = 2\hat{i} - \hat{j} + 3\hat{k}$, then find (i) $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d})$, (ii) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

Solution: We know that,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & -3 & 1 \end{vmatrix}$$

$$= \hat{i}(1 + 3) - \hat{j}(2 - 1) + \hat{k}(-6 - 1)$$

$$= 4\hat{i} - \hat{j} - 7\hat{k}$$

Also $\vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix}$

$$= \hat{i}(3 + 1) - \hat{j}(-3 - 2) + \hat{k}(1 - 2)$$

$$= 4\hat{i} + 5\hat{j} - \hat{k}$$

Now, i) $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (4\hat{i} - \hat{j} - 7\hat{k}).(4\hat{i} + 5\hat{j} - \hat{k})$

$$= 4 \times 4 + (-1).5 + (-7).(-1)$$

$$= 18$$

ii) We know that,

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

Now, $[\vec{a}\ \vec{b}\ \vec{d}] = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -3 & 1 \\ 2 & -1 & 3 \end{vmatrix}$

$= 2(-9 + 1) - 1(3 - 2) + 1(-1 + 6)$

$= -12$

and $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -3 & 1 \\ -1 & 1 & 1 \end{vmatrix}$

$= 2(-3 - 1) - 1(1 + 1) + 1(1 - 3)$

$= -12$

$\therefore (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = -12(-\hat{i} + \hat{j} + \hat{k}) - (-12)(2\hat{i} - \hat{j} + 3\hat{k})$

$= 12\hat{i} - 12\hat{j} - 12\hat{k} + 24\hat{i} - 12\hat{j} + 36\hat{k}$

$= 36\hat{i} - 24\hat{j} + 24\hat{k}$



CHECK YOUR PROGRESS

Q.1: If $\vec{a} = 7\hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = 2\hat{i} + 8\hat{j} + \hat{k}$, find $[\vec{a}\ \vec{b}\ \vec{c}]$.

Q.2: If $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{c} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{d} = 3\hat{i} + \hat{j} + \hat{2}$, and then evalute.

i) $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$ ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$



14.6 LET US SUM UP

- Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors.then the product $\vec{a}.(\vec{b}\times\vec{c})$ is called the scalar triple product of $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$.
- The scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ reprsents numerically the volume of a rectangular parallelopiped having $\vec{a}$, $\vec{b}$, $\vec{c}$ as coterminous edges.
- Let, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

  then, $\vec{a}.(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.
- The scalar product $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$ is called scalar product of four vectors and $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}$
- The vector product $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ is called vector product of four vectors.
- $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$



14.7 FURTHER READING

1) Kishan, Hari. *Vector Algebra and Calculus*. New Delhi: Atlantic Publishers & Distributors (P) Ltd.
2) Narayan, Shanti. *A Textbook of Vector Calculus*.
3) Pandey, R. K. *Vector Analysis*. New Delhi: Discovery Publishing House.
4) Sharma, Anil Kumar. *Textbook of Vector Calculus*. New Delhi: Discovery Publishing House.
5) Tallack, J. C. *Introduction to Vector Analysis*. Cambridge University Press.
6) *Vector Algebra*. New Delhi: Laxmi Publications(P) Ltd.

14.8 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 7 & -2 & 3 \\ 1 & -2 & 2 \\ 2 & 8 & 0 \end{vmatrix}$

$= 7(0 - 16) + 2(0 - 4) + 3(8 + 4)$

$= -112 - 8 + 36$

$= -84$

Ans. to Q. No. 2: i) –38, ii) $-\vec{17}\,\hat{i} - 19\,\hat{j} - 8\,\hat{k}$

14.9 MODEL QUESTIONS

Q 1: Evaluate $\vec{a}.(\vec{b} \times \vec{c})$ if $\vec{a} = 2\hat{i} - 3\hat{j}$, $\vec{b} = \hat{i} + \hat{j} - \hat{k}$, and $\vec{c} = 3\hat{i} - \hat{k}$.

Q 2: Find the volume of the paralellopiped three of whose coterminuous edges are represented by the vectors $\hat{i} + \hat{j} + \hat{k}$, $\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - \hat{k}$.

Q 3: If $\vec{a} = \hat{i} - 2\hat{j} - 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$, and $\vec{c} = \hat{i} + 3\hat{j} - \hat{k}$, find

(i) $\vec{a} \times (\vec{b} \times \vec{c})$ (ii) $(\vec{a} \times \vec{b}) \times \vec{c}$ (iii) $(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})$.

Q 4: Prove that i) $\{(\vec{a} + \vec{b} + \vec{c}) \times (\vec{b} \times \vec{c})\}.c = \vec{a}.(\vec{b} \times \vec{c})$

ii) $(\vec{a} \times \vec{b}).(\vec{b} \times \vec{c}) \times (\vec{a} \times \vec{c}) = 2\vec{a}.\vec{b} \times \vec{c}$

Q 5: Prove that i) $(\vec{a} \times \vec{b}).(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = (\vec{a}.\vec{b} \times \vec{c})$

ii) $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}) = (\vec{a}.\vec{b} \times \vec{c})\vec{a}$

Q 6: If $\vec{a} = 2\hat{i} - 3\hat{j} - \hat{k}$, $\vec{b} = \hat{i} + 4\hat{j} - 2\hat{k}$, find $(\vec{a} \times \vec{b}) \times (\vec{a} - \vec{b})$.

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UNIT 15: VECTOR CALCULUS

UNIT STUCTURE

15.1 Learning Objectives

15.2 Introduction

15.3 Scalar and Vector Point Function

15.4 Vector Differential Operator (del)

15.5 Gradient, Divergence and Curl and their Identities

15.6 Lapalacian Operator

15.7 Let Us Sum Up

15.8 Further Reading

15.9 Answers to Check Your Progress

15.10 Model Questions

15.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define scalar and vector point function
- define vector differential operator(del)
- define gradient, divergence and curl and their identities
- know about Lapalacian operator.

15.2 INTRODUCTION

In unit 12, we have already discussed about the concept of vectors.We also discuss the product of vectors in unit 13 and 14. In this unit,we deal with the theory of vector calculus. With the help of a standard vector differential operator, we introduce concepts like gradient of a scalar valued function, divergence and curl of a vector valued function. In this unit, we also discuss the properties of gradient, divergence and curl.

15.3 SCALAR AND VECTOR POINT FUNCTION

If corresponding to each point p of a region R in space,there corresponds a scalar denoted by $\phi(p)$, then ϕ is said to be a scalar point function for the region R.

If the coordinates of p be (x, y, z), then $\phi(p) = \phi$.

Examples:

a) The temperature $\phi(p)$ at any point p of a fluid occupying a certain region is a scalar point function.

b) The density $\phi(p)$ at any point p of a body occupying a certain region is a scalar point function.

If corresponding to each point p of a region R in space,there corresponds a vector denoted by $\vec{f}(p)$, then $\vec{f}$ is called a vector point function for the region R.

If the coordinates of p be (x, y, z), then,

$\vec{f}(p) = \vec{f}(x, y, z) = f_1(x, y, z)\hat{i} + f_2(x, y, z)\hat{j} + f_3(x, y, z)\hat{k}$

Examples:

a) The velocity $\vec{f}(p)$ of a particle at any time t occupying the position p in a certain region is a vector point function for that region.

b) The acceleration $\vec{f}(p)$ of a particle at any time t occupying the position p in a certain region is a vector point function for that region.

15.4 VECTOR DIFFERENTIAL OPERATOR (DEL)

The vector differential operator ∇ (read as del or nabla) is defined as $\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ where, $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors along the three rectangular axes OX,OY and OZ.The symbol ∇ (del) is called 'nabla'.

15.5 GRADIENT, DIVERGENCE AND CURL AND THEIR IDENTITIES

Gradient: Let $\phi(x, y, z)$ be a scalar point function and continuously differentiable,then the vector,

$$\nabla\phi \equiv \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

is called the gradient of the scalar function ϕ and is written as grad ϕ.

That is grad $\phi = \nabla\phi$.

Note: 1) grad $\phi = \nabla\phi$ is a vector whose three components are,

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}.$$

2) The operator $\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ acts on ϕ. $\nabla\phi$ may be assumed to be the multiplication of a vector

$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ by a scalar.

3) $\nabla\phi \neq \phi\nabla$. There will be no '.' or '×' between ϕ and ∇.

Properties of gradient.

Property 1: If f and g are any two scalar point functions then,

$\nabla(f + g) = \nabla f + \nabla g$.

Proof: $\nabla(f + g) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(f + g)$

$$= \hat{i}\frac{\partial}{\partial x}(f+g) + \hat{j}\frac{\partial}{\partial y}(f+g) + \hat{k}\frac{\partial}{\partial z}(f+g)$$

$$= \hat{i}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) + \hat{j}\left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right) + \hat{k}\left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) + \left(\hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y} + \hat{k}\frac{\partial g}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)f + \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$\therefore \nabla(f + g) = \nabla f + \nabla g$.

Note: Similarly, we can easily prove that, $\nabla(f - g) = \nabla f - \nabla g$.

The above results can also be expressed as:

grad $(f \pm g)$ = grad f ±grad g

Property 2: If f and g are any two scalar point functions then,

$\nabla fg = f\nabla g + g\nabla$.

Proof: $\nabla fg = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(fg)$

$= \hat{i}\frac{\partial}{\partial x}(fg) + \hat{j}\frac{\partial}{\partial y}(fg) + \hat{k}\frac{\partial}{\partial z}(fg)$

$= \hat{i}\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right) + \hat{j}\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right) + \hat{k}\left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)$

$= \hat{i}f\frac{\partial g}{\partial x} + \hat{i}g\frac{\partial f}{\partial x} + \hat{j}f\frac{\partial g}{\partial y} + \hat{j}g\frac{\partial f}{\partial y} + \hat{k}f\frac{\partial g}{\partial z} + \hat{k}g\frac{\partial f}{\partial z}$

$= f\left(\hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y} + \hat{k}\frac{\partial g}{\partial z}\right) + g\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$

$= \nabla f + \nabla g.$

Note: The above result can also be expressed as:

grad(fg) = f grad g + g grad f

Property 3: Gradient of a constant is zero vector.

Proof: Let $\phi(x, y, z)$ be a constant. Then, $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$, $\frac{\partial \phi}{\partial z}$ are zero.

$\therefore \nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} = \vec{0}.$

ILLUSTRATIVES EXAMPLES:

Example 1: If $\phi = x^2 + y - z - 1$, find grad ϕ at (1, 0, 0).

Solution: grad f = $\nabla\phi$

$= \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$= \hat{i}\frac{\partial}{\partial x}(x^2 + y - z - 1) + \hat{j}\frac{\partial}{\partial y}(x^2 + y - z - 1) + \hat{k}\frac{\partial}{\partial z}(x^2 + y - z - 1)$

$= \hat{i}(2x) + \hat{j}(1) + \hat{k}(-1)$

$\therefore \text{grad}\phi_{(1, 0, 0)} = 2\hat{i} + \hat{j} - \hat{k}.$

Example 2: Find gradϕ if $\phi = xyz$ at (1, 1, 1).

Solution: Given $\phi = xyz$

grad f = $\nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$$= \hat{i}\frac{\partial}{\partial x}(xyz) + \hat{j}\frac{\partial}{\partial y}(xyz) + \hat{k}\frac{\partial}{\partial z}(xyz)$$

$$= \hat{i}(yz) + \hat{j}(xz) + \hat{k}(xy)$$

$\therefore \text{grad}\phi_{(1,1,1)} = \hat{i} + \hat{j} + \hat{k}$.

Example 3: If $\phi = \log(x^2 + y^2 + z^2)$, find $\nabla\phi$.

Solution: Given $\phi = \log(x^2 + y^2 + z^2)$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}\log(x^2 + y^2 + z^2) + \hat{j}\frac{\partial}{\partial y}\log(x^2 + y^2 + z^2) + \hat{k}\frac{\partial}{\partial z}\log(x^2 + y^2 + z^2)$$

$$= \hat{i}\left(\frac{2x}{x^2 + y^2 + z^2}\right) + \hat{j}\left(\frac{2y}{x^2 + y^2 + z^2}\right) + \hat{k}\left(\frac{2z}{x^2 + y^2 + z^2}\right)$$

$$= \frac{2}{x^2 + y^2 + z^2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{2}{x^2 + y^2 + z^2}.\vec{r} \quad (\vec{r} = x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{2\vec{r}}{r^2}$$

Normal to the Surface: Let $\phi(x, y, z)$ be a scalar valued function having continuous partial derivatives.Let $P(x_0, y_0, z_0)$ be any point. Let $\phi(x_0, y_0, z_0) = c$. Then the equation $\phi(x, y, z) = c$ represents a surface. Obviously (x_0, y_0, z_0) lies on this surface.

Along this surface $d\phi = 0$

i.e., $\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$

i.e., $\nabla\phi.d\vec{r} = 0$ where, $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$\therefore \nabla\phi$ is perpendicular to $d\vec{r}$.

$\therefore \nabla\phi$ is normal to all the tangents to the surface at $P(x_0, y_0, z_0)$.

Hence $\nabla\phi$ represents the normal to the surface $\phi(x, y, z) = c$.

Hence the unit normal $\hat{n}$ to the surface $\phi(x, y, z) = c$ is given by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}.$$

Directional Derivatives: Let $\hat{n}$ be a unit vector.The component of the vector $\nabla\phi$ in the direction $\hat{n}$ is given by $\nabla\phi.\hat{n}$ and is called the directional derivative of ϕ in the direction $\hat{n}$.

Hence directional derivative of a scalar function ϕ at a point is defined by the dot product of gradϕ and the unit normal vector through that point.

i.e., directional derivative = grad$\phi\hat{n}$

Example 1: Find the directional derivative of f = xyz at (1, 1, 1) in the direction of $\hat{i}+\hat{j}+\hat{k}$.

Solution: Unit normal vector $\hat{n}$ in the direction of $\hat{i}+\hat{j}+\hat{k}$ is:

$\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$.

Now, gradf = $\nabla f = \hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}$

$= \hat{i}\frac{\partial}{\partial x}(xyz) + \hat{j}\frac{\partial}{\partial y}(xyz) + \hat{k}\frac{\partial}{\partial z}(xyz)$

$= \hat{i}(yz) + \hat{j}(xz) + \hat{k}(xy)$

$\therefore (gradf)_{(1,1,1)} = \hat{i}+\hat{j}+\hat{k}$.

Directional derivatives = grad$\phi\hat{n}$

$= (\hat{i}+\hat{j}+\hat{k}).\ \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

$= \frac{1+1+1}{\sqrt{3}}$

$= \frac{3}{\sqrt{3}}$

$= \sqrt{3}$

Note: 1) Normal derivative of ϕ at a point = $|\text{grad}\phi| = |\nabla\phi|$

2) Unit normal vector to the surface $\phi = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\text{grad}\phi}{|\text{grad}\phi|}$

Example 1: Find the normal derivative of $\phi = xy + yz + zx$ at (-1, 1, 1).

Solution: Normal derivative of ϕ at a point = $|\text{grad}\phi| = |\nabla\phi|$

Given $\phi = xy + yz + zx$

$\therefore \text{grad}\phi = \nabla\phi$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial(xy+yz+zx)}{\partial x} + \hat{j}\frac{\partial(xy+yz+zx)}{\partial y} + \hat{k}\frac{\partial(xy+yz+zx)}{\partial z}$$

$$= \hat{i}(y+z) + \hat{j}(z+x) + \hat{k}(x+y)$$

$\therefore (\text{grad}\phi)_{(-1,\,1,\,1)} = 2\hat{i} + 0\hat{j} + 0\hat{k}$

Normal derivative at (–1, 1, 1) is $|\text{grad}\phi| = \sqrt{2^2+0^2+0^2} = 2$.

Example 2: Find the unit normal vector to the surface $\phi = x^3 - xyz + z^3 - 1$ at the point (1, 1, 1).

Solution: Given, $\phi = x^3 - xyz + z^3 - 1$

$\therefore \text{grad}\phi = \nabla\phi$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial(x^3-xyz+z^3-1)}{\partial x} + \hat{j}\frac{\partial(x^3-xyz+z^3-1)}{\partial y} + \hat{k}\frac{\partial(x^3-xyz+z^3-1)}{\partial z}$$

$$= \hat{i}(3x^2 - yz) + \hat{j}(-xz) + \hat{k}(-xy + 3z^2)$$

$$\therefore (\text{grad}\phi)_{(1,\,1,\,1)} = 2\hat{i} + \hat{j} + 2\hat{k}$$

Unit normal vector to the surface $\phi = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{\text{grad}\phi}{|\text{grad}\phi|}$$

$$= \frac{2\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{4+1+4}}$$

$$= \frac{2\hat{i} - \hat{j} + 2\hat{k}}{3}.$$

Example 3: Find the unit normal vector to the surface of the paraboloid of revolution $z = x^2 + y^2$ at the point (1, 2, 5).

Solution: Given, $\phi = x^2 + y^2 - z$

$\therefore$ grad$\phi = \nabla\phi$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial\left(x^2 + y^2 - z\right)}{\partial x} + \hat{j}\frac{\partial\left(x^2 + y^2 - z\right)}{\partial y} + \hat{k}\frac{\partial\left(x^2 + y^2 - z\right)}{\partial z}$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$(\nabla\phi)_{(1, 2, 5)} = 2\hat{i} + 4\hat{j} - \hat{k}$

$|\nabla\phi| = \sqrt{4 + 16 + 1} = \sqrt{21}$

$\therefore$ unit normal vector $= \hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{21}}$.



CHECK YOUR PROGRESS

Q.1: Find grad ϕ when, ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point (1, –2, –1).

Q.2: If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that (i) $\nabla r = \dfrac{\vec{r}}{r}$, (ii) $\nabla r^n = nr^{n-2}\vec{r}$, where, $r = |\vec{r}|$.

Q.3: Find the directional derivatives of $\phi = x^2yz + 4xz^2$ at (1, 1, 1) in the direction of $\hat{i} + \hat{j} - \hat{k}$.

Q.4: Find the directional derivative of $\phi = 3x^2 + 2y - 3z$ at (1, 1, 1) in the direction of $2\hat{i} + 2\hat{j} - \hat{k}$.

Q.5: Find a unit vector to the surface $x^2 + y^2 - z = 10$ at (1, 1, 1).

Q.6: Find a unit vector to the surface $x^2 + xy + z^2 = 4$ at the point (1, –1, 2).

Divergence: Let, $\vec{F}(x, y, z)$ be a vector function which is continuously differentiable at each point (x, y, z). Let $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ be its components along the coordinates axes.

Then, $\vec{F}(x, y, z) = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$

The divergence of the vector function $\vec{F}$ is defined as:

$$\nabla.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Divergence of a vector function $\vec{F}$ is denoted by $\nabla.\vec{F}$ or div $\vec{F}$.

Note: 1) $\nabla.\vec{F}$ or div $\vec{F}$ is a scalar quantity.

2) div $\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial x} + \frac{\partial F_3}{\partial x}$.

Definition: Solenoidal vector: A vector $\vec{F}$ is said to be solenoidal if its divergence is zero. i.e., if div $\vec{F} = 0$, then $\vec{F}$ is said to be solenoidal. i.e., $\nabla.\vec{F} = 0$.

Curl: The curl of $\vec{F}(x, y, z)$ is defined by

$$\nabla \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Definition: Irrotational: A vector $\vec{F}$ is said to be irrotational if its curl is zero vector. i.e., if curl $\vec{F} = \vec{0}$, then $\vec{F}$ is said to be irrotational.

Note: 1) curl $\vec{F}$ is a vector quantity.

2) If ϕ is a scalar function of x, y, z then, $\nabla^2\phi = (\nabla.\nabla)\phi$.

$$\nabla.(\nabla\phi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

i.e., div(gradϕ) = $\nabla^2\phi$.

15.6 LAPLACIAN OPERATOR

The operator $\nabla^2 \equiv \nabla.\nabla$ is defined as $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ and is called the Laplacian Operator.

Laplacian Operator can be operated on scalars as well as vector point functions.

1) When F is a scalar point function, then, $\nabla^2 F = \frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}$ which is a scalar quantity.

2) When $\vec{F}$ is a vector point function, then, $\nabla^2\vec{F} = \frac{\partial^2 \vec{F}}{\partial x^2}+\frac{\partial^2 \vec{F}}{\partial y^2}+\frac{\partial^2 \vec{F}}{\partial z^2}$ which is a vector quantity.

Laplace Equation: $\nabla^2 F = 0$ or, $\nabla^2\vec{F} = 0$ is called Laplace's Equation.



Pierre – Simon Laplace (1749-1827) was a French Mathematician and Astronomer.

Properties of Divergence:

I) $\nabla.(\vec{F} \pm \vec{G}) = \nabla.\vec{F} \pm \nabla.\vec{G}$ or, $\text{div}(\vec{F} \pm \vec{G}) = \text{div}\,\vec{F} \pm \text{div}\,\vec{G}$

Proof: $\nabla.(\vec{F} \pm \vec{G}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\left(\vec{F}\pm\vec{G}\right)$

$$= \hat{i}.\frac{\partial}{\partial x}\left(\vec{F}\pm\vec{G}\right)+\hat{j}.\frac{\partial}{\partial y}\left(\vec{F}\pm\vec{G}\right)+\hat{k}.\frac{\partial}{\partial z}\left(\vec{F}\pm\vec{G}\right)$$

$$= \left(\hat{i}.\frac{\partial\vec{F}}{\partial x}+\hat{j}.\frac{\partial\vec{F}}{\partial y}+\hat{k}.\frac{\partial\vec{F}}{\partial z}\right)\pm\left(\hat{i}.\frac{\partial\vec{G}}{\partial x}+\hat{j}.\frac{\partial\vec{G}}{\partial y}+\hat{k}.\frac{\partial\vec{G}}{\partial z}\right)$$

$$= \nabla.\vec{F} \pm \nabla.\vec{G}$$

$\therefore \nabla.(\vec{F} \pm \vec{G}) = \nabla.\vec{F} \pm \nabla.\vec{G}$

II) $\nabla.(\phi\vec{F}) = (\nabla\phi).\vec{F} + \phi(\nabla.\vec{F})$

Proof: Let, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Now, $\nabla.(\phi\vec{F}) = \nabla.(\phi F_1\hat{i} + \phi F_2\hat{j} + \phi F_3\hat{k})$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).(\phi F_1\hat{i} + \phi F_2\hat{j} + \phi F_3\hat{k})$$

$$= \frac{\partial}{\partial x}(\phi F_1) + \frac{\partial}{\partial y}(\phi F_2) + \frac{\partial}{\partial z}(\phi F)$$

$$= \frac{\partial \phi}{\partial x}F_1 + \phi\frac{\partial F_1}{\partial x} + \frac{\partial \phi}{\partial y}F_2 + \phi\frac{\partial F_2}{\partial y} + \frac{\partial \phi}{\partial z}F_3 + \phi\frac{\partial F_3}{\partial z}$$

$$= \left(\frac{\partial \phi}{\partial x}F_1 + \frac{\partial \phi}{\partial y}F_2 + \frac{\partial \phi}{\partial z}F_3\right) + \phi\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right).(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) +$$

$$\phi\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= (\nabla\phi).\vec{F} + \phi(\nabla.\vec{F})$$

$$\therefore \nabla.(\phi\vec{F}) = (\nabla\phi).\vec{F} + \phi(\nabla.\vec{F})$$

III) $\nabla\times(\vec{F} \pm \vec{G}) = \nabla\times\vec{F} \pm \nabla\times\vec{G}$ or, $\text{curl}(\vec{F} \pm \vec{G}) = \text{curl}\,\vec{F} \pm \text{curl}\,\vec{G}$

Proof: $\nabla\times(\vec{F} \pm \vec{G}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\vec{F} \pm \vec{G}\right)$

$$= \hat{i}\times\frac{\partial}{\partial x}\left(\vec{F} \pm \vec{G}\right) + \hat{j}\times\frac{\partial}{\partial y}\left(\vec{F} \pm \vec{G}\right) + \hat{k}\times\frac{\partial}{\partial z}\left(\vec{F} \pm \vec{G}\right)$$

$$= \left(\hat{i}\times\frac{\partial\vec{F}}{\partial x} + \hat{j}\times\frac{\partial\vec{F}}{\partial y} + \hat{k}\times\frac{\partial\vec{F}}{\partial z}\right) \pm \left(\hat{i}\times\frac{\partial\vec{G}}{\partial x} + \hat{j}\times\frac{\partial\vec{G}}{\partial y} + \hat{k}\times\frac{\partial\vec{G}}{\partial z}\right)$$

$$= \nabla\times\vec{F} \pm \nabla\times\vec{G}$$

$$\therefore \nabla\times(\vec{F} \pm \vec{G}) = \nabla\times\vec{F} \pm \nabla\times\vec{G}$$

IV) $\nabla\times(\phi\vec{F}) = (\nabla\phi)\times\vec{F} + \phi(\nabla\times\vec{F})$ or, $\text{curl}(\phi\vec{F}) = \text{grad}\phi\times\vec{F} + \phi\,\text{curl}\,\vec{F}$

Proof: Let, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, $\phi\vec{F} = \phi F_1\hat{i} + \phi F_2\hat{j} + \phi F_3\hat{k}$

Now, $\nabla\times(\phi\vec{F}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(\phi F_1\hat{i} + \phi F_2\hat{j} + \phi F_3\hat{k})$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi F_1 & \phi F_2 & \phi F_3 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(\phi F_3)-\frac{\partial}{\partial z}(\phi F_2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(\phi F_1)-\frac{\partial}{\partial x}(\phi F_3)\right] +$$

$$\hat{k}\left[\frac{\partial}{\partial x}(\phi F_2)-\frac{\partial}{\partial y}(\phi F_1)\right]$$

$$= \hat{i}\left[\phi\frac{\partial F_3}{\partial y}+\frac{\partial \phi}{\partial y}F_3-\phi\frac{\partial F_2}{\partial z}-\frac{\partial \phi}{\partial z}F_2\right] + \hat{j}\left[\phi\frac{\partial F_1}{\partial z}+\frac{\partial \phi}{\partial z}F_1-\phi\frac{\partial F_3}{\partial x}-\frac{\partial \phi}{\partial x}F_3\right]$$

$$+ \hat{k}\left[\phi\frac{\partial F_2}{\partial x}+\frac{\partial \phi}{\partial x}F_2-\phi\frac{\partial F_1}{\partial y}-\frac{\partial \phi}{\partial y}F_1\right]$$

$$= \phi\left[\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\hat{i}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\hat{j}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\hat{k}\right] +$$

$$\left[\left(\frac{\partial \phi}{\partial y}F_3-\frac{\partial \phi}{\partial x}F_2\right)\hat{i}+\left(\frac{\partial \phi}{\partial z}F_1-\frac{\partial \phi}{\partial x}F_3\right)\hat{j}+\left(\frac{\partial \phi}{\partial x}F_2-\frac{\partial \phi}{\partial y}F_1\right)\hat{k}\right]$$

$$= \phi\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}+\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}$$

$\therefore \nabla\times(\phi\vec{F}) = (\nabla\phi)\times\vec{F} + \phi(\nabla\times\vec{F})$

V) $\nabla.(\vec{F}\times\vec{G}) = \vec{G}.(\nabla\times\vec{F}) - \vec{F}.(\nabla\times\vec{G})$ or, $\text{div}(\vec{F}\times\vec{G}) = \vec{G}\,\text{curl}\,\vec{F} - \vec{F}.\,\text{curl}\,\vec{G}$

Proof: By definition, we have, $\nabla.\vec{F} = \hat{i}.\frac{\partial \vec{F}}{\partial x}+\hat{j}.\frac{\partial \vec{F}}{\partial y}+\hat{k}.\frac{\partial \vec{F}}{\partial z}$

Hence, $\nabla.(\vec{F}\times\vec{G}) = \hat{i}.\frac{\partial}{\partial x}(\vec{F}\times\vec{G}) + \hat{j}.\frac{\partial}{\partial y}(\vec{F}\times\vec{G}) + \hat{k}.\frac{\partial}{\partial z}(\vec{F}\times\vec{G})$

$$= \hat{i}.\left[\vec{F}\times\frac{\partial \vec{G}}{\partial x}+\frac{\partial \vec{F}}{\partial x}\times\vec{G}\right]+\hat{j}.\left[\vec{F}\times\frac{\partial \vec{G}}{\partial y}+\frac{\partial \vec{F}}{\partial y}\times\vec{G}\right]+\hat{k}.\left[\vec{F}\times\frac{\partial \vec{G}}{\partial z}+\frac{\partial \vec{F}}{\partial z}\times\vec{G}\right]$$

$$= \hat{i}.\vec{F}\times\frac{\partial \vec{G}}{\partial x}+\hat{i}.\frac{\partial \vec{F}}{\partial x}\times\vec{G}+\hat{j}.\vec{F}\times\frac{\partial \vec{G}}{\partial y}+\hat{j}.\frac{\partial \vec{F}}{\partial y}\times\vec{G}+\hat{k}.\vec{F}\times\frac{\partial \vec{G}}{\partial z}+\hat{k}.\frac{\partial \vec{F}}{\partial z}\times\vec{G}$$

$$= -\hat{i}.\frac{\partial \vec{G}}{\partial x}\times\vec{F}+\hat{i}.\frac{\partial \vec{F}}{\partial x}\times\vec{G}-\hat{j}.\frac{\partial \vec{G}}{\partial y}\times\vec{F}+\hat{j}.\frac{\partial \vec{F}}{\partial y}\times\vec{G}-\hat{k}.\frac{\partial \vec{G}}{\partial z}\times\vec{F}+\hat{k}.\frac{\partial \vec{F}}{\partial z}\times\vec{G}$$

By interchanging the dot and cross products, we get,

$$= -\hat{i}\times\frac{\partial\vec{G}}{\partial x}.\vec{F}+\hat{i}\times\frac{\partial\vec{F}}{\partial x}.\vec{G}-\hat{j}\times\frac{\partial\vec{G}}{\partial y}.\vec{F}+\hat{j}\times\frac{\partial\vec{F}}{\partial y}.\vec{G}-\hat{k}\times\frac{\partial\vec{G}}{\partial z}.\vec{F}+\hat{k}\times\frac{\partial\vec{F}}{\partial z}.\vec{G}$$

$$= \left(\hat{i}\times\frac{\partial\vec{F}}{\partial x}+\hat{j}\times\frac{\partial\vec{F}}{\partial y}+\hat{k}\times\frac{\partial\vec{F}}{\partial z}\right).\vec{G}-\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}+\hat{j}\times\frac{\partial\vec{G}}{\partial y}+\hat{k}\times\frac{\partial\vec{G}}{\partial z}\right).\vec{F}$$

$$= \text{curl}\,\vec{F}\,.\,\vec{G} - \text{curl}\,\vec{G}\,.\,\vec{F}$$

$$\therefore \text{div}(\vec{F}\times\vec{G}) = \text{curl}\,\vec{F}\,.\,\vec{G} - \text{curl}\,\vec{G}\,.\,\vec{F}$$

VI) $\nabla\times(\nabla\times\vec{F}) = \nabla(\nabla.\vec{F}) - \nabla^2\vec{F}$ or, $\text{curlcurl}\,\vec{F} = \text{graddiv}\,\vec{F} - \text{Laplacian}\,\vec{F}$

Proof: Let, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\nabla\times\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(F_1\hat{i}+F_2\hat{j}+F_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+\hat{j}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+\hat{k}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$\text{curl}(\text{curl}\,\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y} \end{vmatrix}$$

$$= \sum\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\right]\hat{i}$$

$$= \sum\left[\frac{\partial^2 F_2}{\partial y\partial x}+\frac{\partial^2 F_3}{\partial z\partial x}-\left(\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\right]\hat{i}$$

$$= \sum\left[\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_2}{\partial y\partial x}+\frac{\partial^2 F_3}{\partial z\partial x}\right)-\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\right]\hat{i}$$

$$= \sum\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F_1\right]\hat{i}$$

$$= \sum\left[\frac{\partial}{\partial x}\left(\nabla.\vec{F}\right)-\nabla^2 F_1\right]\hat{i}$$

$$= \left[\hat{i}\frac{\partial}{\partial x}\left(\nabla.\vec{F}\right)+\hat{j}\frac{\partial}{\partial y}\left(\nabla.\vec{F}\right)+\hat{k}\frac{\partial}{\partial z}\left(\nabla.\vec{F}\right)\right]-\nabla^2\left[F_1\hat{i}+F_2\hat{j}+F_3\hat{k}\right]$$

$$= \nabla.(\nabla.\vec{F})-\nabla^2\vec{F}$$

$\therefore \nabla\times(\nabla\times\vec{F}) = \nabla.(\nabla.\vec{F})-\nabla^2\vec{F}$

VII) $\nabla.(\nabla\times\vec{F}) = 0$ or, $\text{divcurl}\,\vec{F} = 0$

Proof: Let, $\vec{F} = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$

$$\nabla\times\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(F_1\hat{i}+F_2\hat{j}+F_3\hat{k})$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+\hat{j}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+\hat{k}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

Now, $\text{divcurl}\,\vec{F} = \nabla.(\nabla\times\vec{F})$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\partial y}-\frac{\partial^2 F_2}{\partial x\partial z}+\frac{\partial^2 F_1}{\partial y\partial z}-\frac{\partial^2 F_3}{\partial y\partial x}+\frac{\partial^2 F_2}{\partial z\partial x}-\frac{\partial^2 F_1}{\partial z\partial y}$$

$= 0$

$\therefore \text{divcurl}\,\vec{F} = 0$

Important properties (without proof):

VIII) $\nabla(\vec{F}.\vec{G}) = \vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})+(\vec{F}.\nabla)\vec{G}+(\vec{G}.\nabla)\vec{F}$ or,

$\text{grad}(\vec{F}.\vec{G}) = \vec{F}\times\text{curl}\,\vec{G}+\vec{G}\times\text{curl}\,\vec{F}+(\vec{F}.\nabla)\vec{G}+(\vec{G}.\nabla)\vec{F}$

IX) $\nabla\times(\vec{F}\times\vec{G}) = \vec{F}(\nabla.\vec{G})-\vec{G}(\nabla.\vec{F})+(\vec{G}.\nabla)\vec{F}+(\vec{F}.\nabla)\vec{G}$ or,

$\text{curl}(\vec{F}\times\vec{G}) = \vec{F}\,\text{div}\,\vec{G}-\vec{G}\,\text{div}\,\vec{F}+(\vec{G}.\nabla)\vec{F}+(\vec{F}.\nabla)v$

ILLUSTRATIVE EXAMPLES:

Example1: If $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$, then find $\nabla.\vec{F}$ and $\nabla \times \vec{F}$.

Solution: $\nabla.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^2\hat{i} + y^2\hat{j} + z^2\hat{k})$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$= 2(x + y + z)$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \vec{0}$$

Example 2: Find the divergence and curl of the vector point function $xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$.

Solution: Let, $\vec{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$

$\text{div}\vec{F} = \nabla.\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k})$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) - \frac{\partial}{\partial z}(3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

$\text{curl}\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(-3yz^2\right)-\frac{\partial}{\partial z}\left(2x^2yz\right)\right]-\hat{j}\left[\frac{\partial}{\partial x}\left(-3yz^2\right)-\frac{\partial}{\partial z}\left(xy^2\right)\right]+$$

$$\hat{k}\left[\frac{\partial}{\partial x}\left(2x^2yz\right)-\frac{\partial}{\partial y}\left(xy^2\right)\right]$$

$$= \hat{i}\,(-3z^2 - 2x^2y) - \hat{j}\,(0) + \hat{k}\,(4xyz - 2xy)$$

Example 3: If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that div $\vec{r} = 3$ and curl $\vec{r} = \vec{0}$.

Solution: $\text{div}\,\vec{F} = \nabla.\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z)-\frac{\partial}{\partial z}(y)\right]-\hat{j}\left[\frac{\partial}{\partial x}(z)-\frac{\partial}{\partial z}(x)\right]+\hat{k}\left[\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(x)\right]$$

$$= \hat{i}\,(0 - 0) - \hat{j}\,(0 - 0) + \hat{k}\,(0 - 0) = 0$$

Example 4: If $\vec{F} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, find curl $\vec{F}$ at (1, –1, 1).

Solution: $\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(2yz^4\right)-\frac{\partial}{\partial z}\left(-2x^2yz\right)\right]-\hat{j}\left[\frac{\partial}{\partial x}\left(2yz^4\right)-\frac{\partial}{\partial z}\left(xz^3\right)\right]+$$

$$\hat{k}\left[\frac{\partial}{\partial x}\left(-2x^2yz\right)-\frac{\partial}{\partial y}\left(xz^3\right)\right]$$

$$= \hat{i}\,(2z^4 + 2x^2y) + \hat{j}\,(3z^2x) + \hat{k}\,(-4xyz)$$

curl $\vec{F}$ at (1, –1, 1) = $\hat{i}(0) + 3\hat{j} + 4\hat{k} = 3\hat{j} + 4\hat{k}$

Example 5: If $\vec{F} = x^2y\hat{i} + y^2z\hat{j} + z^2x\hat{k}$, find curlcurl $\vec{F}$.

Solution: $\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2x \end{vmatrix}$

$$= i\left[\frac{\partial}{\partial y}(xz^2) - \frac{\partial}{\partial z}(y^2z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(xz^2) - \frac{\partial}{\partial z}(x^2y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2z) - \frac{\partial}{\partial y}(x^2z)\right]$$

$$= \hat{i}(0 - y^2) - \hat{j}(z^2 - 0) + \hat{k}(0 - x^2)$$

$$= -y^2\hat{i} - z^2\hat{j} - x^2\hat{k}$$

$$\text{curlcurl}\vec{F} = \nabla \times \text{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & -z^2 & -z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-x^2) - \frac{\partial}{\partial z}(-z^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial z}(-y^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-z^2) - \frac{\partial}{\partial y}(-y^2)\right]$$

$$= \hat{i}(0 + 2z) - \hat{j}(-2x) - \hat{k}(-2y)$$

$$= 2x\hat{i} + 2x\hat{j} + 2y\hat{k}$$

Example 6: Prove that curl(gradϕ) = $\vec{0}$

Solution: curl(gradϕ) = curl($\nabla\phi$) = $\nabla \times (\nabla\phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial y} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] - \hat{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right] + \hat{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$$= \vec{0} + \vec{0} + \vec{0} = \vec{0}.$$

Example 7: Check whether the vector $x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ is a solenoidal at the point (1, 1, 2)?

Solution: Let, $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

$$\nabla.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^2\hat{i} + y^2\hat{j} + z^2\hat{k})$$

$$= \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z}$$

$$= 2x + 2y + 2z$$

$$(\nabla.\vec{F})_{(1,\,1,\,2)} = 2 + 2 + 4 = 8 \neq 0$$

$\therefore$ The given vector is not solenoidal at the point (1, 1, 2).



CHECK YOUR PROGRESS

Q.7: Find the divergence and curl of the vector:

$\vec{V} = xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$ at the point $(2, -1, 1)$.

Q.8: Find $\text{div}\,\vec{F}$ and $\text{curl}\,\vec{F}$ where, $\vec{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$.

Q.9: Determine the constant a, so that the vector:

$\vec{F} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}$ is solenoidal.



15.7 LET US SUM UP

- The vector differential operator ∇ (read as del or nabla) is defined as $\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ where, $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors along the three rectangular axes OX, OY and OZ.
- Let $\phi(x, y, z)$ be a scalar point function and continuously differentiable, then the vector $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$ is called the gradient of ϕ.

- Properties of gradient:
  i) grad(f ± g) = grad f ± gradg g
  ii) If f and g are any two scalar point functions then, $\nabla fg = f\nabla g + g\nabla f$.
  iii) Gradient of a constant is zero.
- The divergence of the vector function:
  $\vec{F}(x, y, z) = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$ is defined as:

$$\nabla.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial x} + \hat{k}\frac{\partial}{\partial x}\right).\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial x} + \frac{\partial F_3}{\partial x}$$

- A vector $\vec{F}$ is said to be solenoidal if its divergence is zero.
- The curl of $\vec{F}(x, y, z)$ is defined by

$$\nabla \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

- A vector $\vec{F}$ is said to be irrotational if its curl is zero vector.
- Properties related to divergence and curl
  i) $\text{div}(\vec{F} \pm \vec{G}) = \text{div}\vec{F} \pm \text{div}\vec{G}$
  ii) $\nabla.(\phi\vec{F}) = (\nabla\phi).\vec{F} + \phi(\nabla.\vec{F})$
  iii) $\text{curl}(\vec{F} \pm \vec{G}) = \text{curl}\vec{F} \pm \text{curl}\vec{G}$
  iv) $\text{curl}(\phi\vec{F}) = \text{grad}\phi \times \vec{F} + \phi\text{curl}\vec{F}\ (\nabla \times \vec{F})$
  v) $\text{div}(\vec{F} \times \vec{G}) = \vec{G}.\text{curl}\vec{F} - \vec{F}.\text{curl}\vec{G}$
  vi) $\text{curlcurl}\vec{F} = \text{graddiv}\vec{F} - \text{Laplacain}\vec{F}$
  vii) $\text{divcurl}\vec{F} = 0$

viii) $\text{grad}(\vec{F}\cdot\vec{G}) = \vec{F}\times\text{curl}\,\vec{G} + \vec{G}\times\text{curl}\,\vec{F} + (\vec{F}\cdot\nabla)\vec{G} + (\vec{G}\cdot\nabla)\vec{F}$

ix) $\text{curl}(\vec{F}\times\vec{G}) = \vec{F}\,\text{div}\,\vec{G} - \vec{G}\,\text{div}\,\vec{F} + (\vec{G}\cdot\nabla)\vec{F} + (\vec{F}\cdot\nabla)\vec{G}$



15.8 FURTHER READING

1) Kishan, Hari. *Vector Algebra and Calculus*. New Delhi: Atlantic Publishers & Distributors (P) Ltd.
2) Narayan, Shanti. *A Textbook of Vector Calculus*.
3) Pandey, R. K. *Vector Analysis*. New Delhi: Discovery Publishing House.
4) Sharma, Anil Kumar. *Textbook of Vector Calculus*. New Delhi: Discovery Publishing House.
5) Tallack, J. C. *Introduction to Vector Analysis*. Cambridge University Press.
6) *Vector Algebra*. New Delhi: Laxmi Publications(P) Ltd.



15.9 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1: $\text{grad}\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$\text{grad}\phi_{(1,-2,-1)} = -12\hat{i} - 9\hat{j} - 16\hat{k}$ at $(1, -2, -1)$

Ans. to Q. No. 2: Given $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

i.e., $r^2 = x^2 + y^2 + z^2$ ... ...(1)

Now $\nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)r$

$$= \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$ ... ...(2)

From (1), we get, $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$$2r\frac{\partial r}{\partial x} = 2y \Rightarrow \frac{\partial r}{\partial x} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial x} = 2z \Rightarrow \frac{\partial r}{\partial x} = \frac{z}{r}$$

From (2), we get, $\nabla r = \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r}$$

$\therefore \nabla r = \frac{\vec{r}}{r}$

ii) $\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n$

$$= \hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}$$

$$= \hat{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \hat{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \hat{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left(\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right)$$

$$= nr^{n-1}\nabla r$$

$$= nr^{n-1}\frac{\vec{r}}{r} \qquad \text{From (i)}$$

$\therefore \nabla r^n = nr^{n-2}\vec{r}$

Ans. to Q. No. 3: Unit normal vector $\hat{n}$ in the direction of $\hat{i} + \hat{j} - \hat{k}$ is $\frac{\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}$.

Now $\text{grad}\phi = \nabla\phi = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$$= \hat{i}\frac{\partial}{\partial x}\left(x^2yz + 4xz^2\right) + \hat{j}\frac{\partial}{\partial y}\left(x^2yz + 4xz^2\right) + \hat{k}\frac{\partial}{\partial z}\left(x^2yz + 4xz^2\right)$$

$$= \hat{i}\,(2xyz + 4z^2) + \hat{j}\,(x^2z) + \hat{k}\,(x^2y + 8xz)$$

$\therefore (\text{grad}\phi)_{(1,1,1)} = \hat{i}(2+4) + \hat{j}(1) + \hat{k}(1+8) = 6\hat{i} + \hat{j} + 9\hat{k}$

Directional derivative = $\text{grad}\phi . \hat{n}$

$$= (6\hat{i} + \hat{j} + 9\hat{k}) . \frac{\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}$$

$$= \frac{6 + 1 - 9}{\sqrt{3}}$$

$$= \frac{-2}{\sqrt{3}}$$

Ans. to Q. No. 4: Do yourself.

Ans. to Q. No. 5: Given $\phi = x^2 + y^2 - z - 10$

$$\therefore \text{grad}\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial\left(x^2 + y^2 - z - 10\right)}{\partial x} + \hat{j}\frac{\partial\left(x^2 + y^2 - z - 10\right)}{\partial y} + \hat{k}\frac{\partial\left(x^2 + y^2 - z - 10\right)}{\partial z}$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$(\nabla\phi)_{(1,1,1)} = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$|\nabla\phi| = \sqrt{4 + 4 + 1} = 3$

$$\therefore \text{unit normal vector} = \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}.$$

Ans. to Q. No. 6: Do Yourself.

Ans. to Q. No. 7: $\text{div}\,\vec{V} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$

$= yz + 3x^2 + 2xz - y^2$

$\therefore \text{div}\,\vec{V} = -1 + 12 + 4 - 1 = 14$ at $(2, -1, 1)$.

$$\text{curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= \hat{i}(-2yz - 0) + \hat{j}(xy - z^2) + \hat{k}(6xy - xz)$$

$\therefore \text{curl}\,\vec{V} = 2\hat{i} - 3\hat{j} - 14\hat{k}$ at $(2, -1, 1)$.

Ans. to Q. No. 8: Let, $\phi = x^3 + y^3 + z^3 - 3xyz$, then,

$$\vec{F} = \text{grad}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$$

$$= (3x^2 - 3yz)\hat{i} + (3y^2 - 3zx)\hat{j} + 3(z^2 - 3xy)\hat{k}$$

$$\therefore \text{div}\vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial z}(z^2 - 3xy)$$

$$= 6x + 6y + 6z$$

$$= 6(x + y + z)$$

$$\text{and curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix}$$

$$= \hat{i}(-3x + 3x) + \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z)$$

$$= \vec{0}$$

Ans. to Q. No. 9: The vector $\vec{F}$ is solenoidal if $\text{div}\vec{F}$.

$$\text{Now, div}\vec{F} = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + az)$$

$$= 1 + 1 + a$$

$$= 2 + a$$

$$\therefore \text{div}\vec{F} = 0$$

$$\Rightarrow 2 + a = 0$$

$$\Rightarrow a = -2.$$



15.10 MODEL QUESTIONS

Q.1: If $\vec{F} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, find find $\nabla.\vec{F}$.

Q.2: If $\phi = 2x^3y^2z^4$, find div(gradϕ).

Q.3: Find $\text{div}\vec{F}$ and $\text{curl}\vec{F}$ for the vector $xy^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$ at (1, –1, 1).

Q.4: If $\vec{F} = 3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}$ and $\phi = 3x^2 - yz$, find

i) $\nabla.\vec{F}$

ii) $\nabla.(\phi\vec{F})$ at (1, –1, 1).

Q.5: If $\phi = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$, find $\nabla^2\phi$.

Q.6: Prove that the vector $\vec{F} = (x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k}$ is solenoidal.

Q.7: Prove that the vector $\vec{F} = (\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x - y)\hat{k}$ is irrotational.

*** ***** ***



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Learner's Feedback on Course

Dear Learner,

Regarding the course as mentioned above, we would like to know your opinions and comments so as to improve the quality of self learning materials in future. Please respond to the following statements by ticking the number you feel most reflect your opinion. After completion of the additional comments, please detach the page and send/mail the same to us at the address given below.

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1) Approximately how many hours did you spend for studying the units in the course? ☐☐☐

2) Please give your opinions (by ✓ mark) to the following items based on your reading of the block:

Sl. No.	Statements	Strongly Agree	Agree	Neither Agree nor Disagree	Disagree	Strongly Disagree
I)	The SLMs of the course fulfil the learning objectives set out in the beginning of each unit					
II)	The units of the SLM could satisfy my academic needs and expectations					
III)	The Unit writers have excellent knowledge about the course contents					
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V)	Ample opportunity for participation in the activities provided in the units					
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VII)	Quality of content is engaging, relevant, and up-to-date					
VIII)	The self check questions are very helpful					
IX)	The Possible/model questions and the answers to check my progress have benefited me a lot					

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1) Which aspects of the SLM, according to you, worked well?

.....

.....

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