GMAS601
Exam Code : MAP6A

MECHANICS AND ELEMENTS OF STATISTICS

SEMESTER - VI

MATHEMATICS

Block - 1



KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

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COURSE INTRODUCTION

'Mechanics and Elements of Statistics' is the sixth course of the programme B. A Mathematics (pass).

The course is divided into two blocks. The first block consists of six units and second block consists of eight units.

Block 1 deals with the basic concept of Mechanics. Concept of force and its typess will be discussed in this block. Lami's theorem will introduced in this block.Concept of simple harmonic motion will also be included in this block.

Block 2 deals with the basic concept of Elements of Statistics. Measures of central tendency and Measures of dispersion will be introduced in this block. Concept of probability ,random variable and probability distribution will also covered in this block.

BLOCK INTRODUCTION

This is the first block of the course 'Mechanics and Elements of Statistics'. The block consists of six units.

In unit 1, We discuss force and its types.

In unit 2,We deals with Lami's theorem.

In unit 3,,We discuss moments and couples.

In unit 4,,We deals with friction and properties of friction.

In unit 5,We discuss Kinematics of a particle ,rectilinear motion under constant acceleration and Newton's Laws of Motion

In unit 6,we deals with simple harmonic motion, geometrical representation of simple harmonic motion and motion of a simple pendulum

While going through a unit, you will notice some along-side boxes, which have been included to help you know some of the difficult, unseen terms. At the end of each section, you will get "CHECK YOUR PROGRESS" questions. These have been designed to self-check your progress of study. It will be better if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with "ANSWERS TO CHECK YOUR PPROGRESS" given at the end of each unit.

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UNIT 1: FORCES

UNIT STRUCTURE

1.1 Learning Objectives

1.2 Introduction

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1.4 Types of Forces

1.5 Magnitude and direction of the resultant of two forces acting on a particle

1.6 Let Us Sum Up

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1.8 Answers To Check Your Progress

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1.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- forces
- types of forces
- magnitude and direction of the resultant of the forces acting on a particle

1.2 INTRODUCTION

Mechanics, the oldest of all physical sciences, deals with the conditions of rest or motion of the objects around us. Mechanics is divided into two branches, viz., statics and Dynamics. Statics may be called the science of rest or equilibrium of bodies, whereas Dynamics may be called the science of motion of bodies. The most fundamental concept in statics as well as in Dynamics is force. In this unit, we will study force and its different types. Also, we will deduce the magnitude and direction of the resultant of the forces acting on a particle.

1.3 FORCES

A force is that which changes or tends to change the state of rest of a body or of its uniform motion in a straight line

A force has three characteristics, viz., (i) a magnitude, (ii) a direction, (iii) a point of application.

A force has both magnitude and direction. So it is a vector quantity.

Let $\overrightarrow{AB}$ denote a force $\vec{F}$. It is acting on a certain body. The length

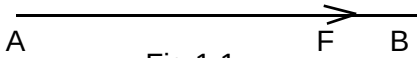


Fig 1.1

AB is the magnitude of the force. The arrowhead denotes the direction of the force. The initial point A of the line segment AB represents the point of application of the force. The line along the direction of the force is called its line of action.

Units of measurements

There are different systems of units for measuring physical quantities, viz., (i) CGS (centimentre-gram-second) system, (ii) MKS (metre-kilogram-second) system, (iii) FPS (foot-pound-second) system.

MKS system is nothing but an extension of CGS system. It is used throughout the world nowadays.

The different systems are shown below

	Mass	Length	Time	Force
CGS	gram (g)	Centimetre (cm)	second (s)	Dyne
MKS	kilogram (kg)	metre (m)	Second (s)	Newton (N)
FPS	pound (lb)	foot (ft)	Second (s)	Poundal (lb. wt.)

1.4 TYPES OF FORCES

(i) Attraction and Repulsion

Pull or push without actual contact of the body or without any visible medium is called attraction or repulsion respectively.

Examples : attraction of a body by earth, attraction or repulsion of a magnet by another.

The force with which the earth attracts a body is called the weight

of the body.

(ii) Tension and Thrust

If a body is pulled by means of a string, the force exerted throughout the string is called tension.

If a body is pushed by means of a rod, the force exerted throughout the rod is called thrust.

(iii) Action and Reaction

If a body exerts a force upon another body, the second body also exerts a force on the first body. These two forces are equal in magnitude, but opposite in direction. Moreover they have a common line of action. Thus, action and reaction are equal and opposite.

(iv) Friction

If a body slides or rolls on another, one or both being rough, there is always a force which tends to resist motion. Friction is the force which resists the sliding of one body over another.

Parallelogram of forces

If two or more forces act on a particle, their total effect is always equivalent to a single force. This single force is called the resultant of the forces. If two forces are acting at a point, their resultant can be obtained by the draw of parallelogram of forces, which is the foundamental theorem of statics.

Law of parallelogram of forces

If two forces acting at a point are represented in magnitude and direction by the sides of a parallelogram drawn from that point, their resultant is represented in magnitude and direction by the diagonal of the parallelogram drawn from that point.

Thus, if the forces $\vec{P}$ and $\vec{Q}$ acting at a point O of a body be represented in magnitude and direction by $\vec{OA}$ and $\vec{OB}$ respectively, their resultant $\vec{R}$ will be represented by the diagonal $\vec{OC}$ drawn through O. Thus, we have

$\vec{OC} = \vec{OA} + \vec{AC} = \vec{OA} + \vec{OB}$ **Q** $\vec{AC} = \vec{OB}$

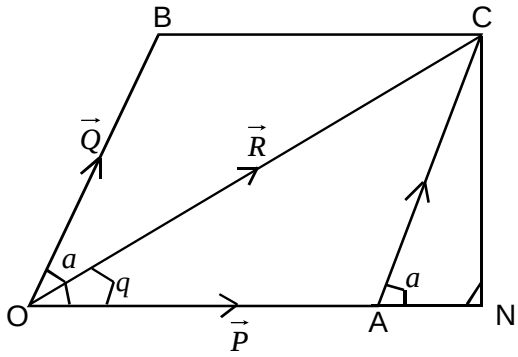


Fig 1.2

1.5 MAGNITUDE AND DIRECTION OF THE RESULTANT OF TWO FORCES ACTING AT A POINT

Let the forces P and Q act at a point O of a body. Let a be the angle between two forces.

Let $\vec{OA}$ and $\vec{OB}$ represent the forces P and Q. We complete the parallelogram OACB.

By the parallelogram law of forces, the resultant R is represented by the diagenal $\vec{OC}$ in magnitude and direction. Let q be the angle made by the line of action of the resultant with that of the force P.

Case I a is an acute angle

From C, we draw CN perpendicular on OA produced.

From the figure, we have

$\angle CAN = \angle BOA = a$

From the right angled triangle CNA,

$$\sin \angle CAN = \frac{CN}{AC}$$

$$\cos \angle CAN = \frac{AN}{AC}$$

$$\therefore CN = AC \sin \angle CAN = OB \sin \angle CAN$$

$$\therefore CN = Q \sin a$$

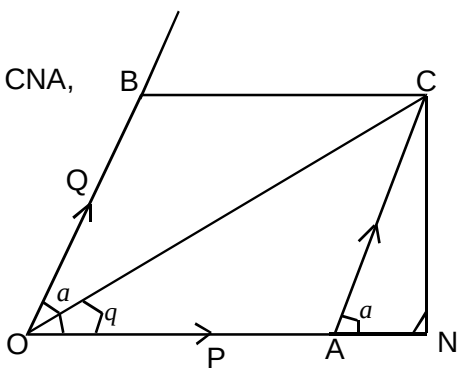


Fig 1.3

$AN = AC\cos\angle CAN = OB\cos\angle CAN$

$\therefore AN = Q\cos a$

$\therefore$ ON = OA + AN = P + Q cos a

Now, from the right angled triangle ONC

$OC^2 = ON^2 + CN^2$

or, $R^2 = (P + Q\cos a)^2 + Q^2\sin^2 a$

or, $R^2 = P^2 + 2PQ\cos a + Q^2\sin^2 a$

$\therefore R^2 = P^2 + 2PQ\cos a + Q^2$ . . . (1) $(\cos^2 a + \sin^2 a = 1)$

From (1), we get the magnitude of the resultant R.

Again, from the right angled triangle ONC,

$\cos q = \frac{ON}{OC} = \frac{P + Q\cos a}{R}$ i.e. $R\cos q = P + Q\cos a$

$\sin q = \frac{CN}{OC} = \frac{Q\sin a}{R}$ i.e. $\therefore R\sin q = Q\sin a$ . . . (2)

From (2), we get the direction of the resultant R.

Also from (2), $\tan q = \frac{q\sin a}{P + Q\cos a}$

Case 2 a is an obtuse angle

From C, we draw CN perpendicular on OA.

From the figure, we have $\angle CAN = 180^0 - a$

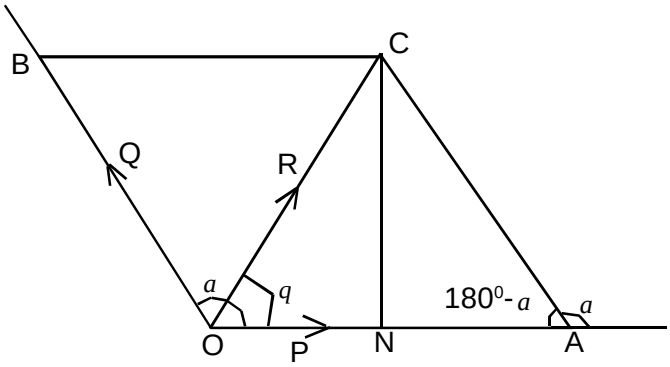


Fig 1.4

From the right angled triangle CNA, $\sin\angle CAN = \frac{CN}{AC}$

$\cos\angle CAN = \frac{AN}{AC}$

$\therefore CN = AC\sin\angle CAN = OB\sin(180^0 - a) = Q\sin a$

$AN = AC\cos\angle CAN = OB\cos(180^0 - a) = -Q\cos a$

$\therefore\ ON = OA - NA = P - (-Q\cos a) = P + Q\cos a$

Now from the right angled triangle ONC.

$OC^2 = ON^2 + CN^2$

or $R^2 = (P + Q\cos a)^2 + Q^2 \sin 2a$

$\therefore\ R^2 = P^2 + 2PQ\cos a + Q^2$ . . . (1)

From (1), we get the magnitude of the resultant R.

Again, from the right angled triangle ONC,

$\cos q = \frac{ON}{OC} = \frac{P + Q\cos a}{R}$ i.e. $R\cos q = P + Q\cos a$ . . . (2)

$\sin q = \frac{CN}{OC} = \frac{Q\sin a}{R}$ i.e. $R\sin q = Q\sin a$

From (2), we get the direction of the resultant R.

Also from (2), $\tan q = \frac{q\sin a}{P + Q\cos a}$

Corollaries

1. If $a = 0$ i.e, if the forces act in the same straight line in the same direction, then from (1) we have

$R^2 = P^2 + 2PQ\cos 0 + Q^2 = P^2 + 2PQ + Q^2$

$= (P + Q)^2$

$\therefore\ R = P + Q$

i.e. the magnitude of the resultant = the sum of the magnitude of the forces.

In this case, the magnitude of the resultant is greatest.

2. If $a = 180^0$, i.e., if the forces act in the same straight line in the opposite direction, them from (1), we have

$R^2 = P^2 + 2PQ\cos 180^0 + Q^2$

$= P^2 - 2PQ + Q^2 = (P \sim Q)^2$

$\therefore\ R = P \sim Q$

i.e. the magnitude of the resultant = the difference of the magnitude of the forces.

In this case, the magnitude of the resultant is least.

3. If P = Q, i.e., if the magnitudes of the forces are equal, then from (1), we have

$$R^2 = P^2 + 2P^2\cos a + P^2$$

$$= 2P^2(1+\cos a) = 4P^2\cos^2\frac{a}{2}$$

$$\therefore\ R = 2P\cos\frac{a}{2}$$

Again, from (3), we have

$$\tan q = \frac{Q\sin a}{Q+Q\cos a} = \frac{\sin a}{1+\cos a}$$

$$= \frac{2\sin\frac{a}{2}\cos\frac{a}{2}}{2\cos^2\frac{a}{2}}$$

$$= \tan\frac{a}{2}$$

i.e. $q = \frac{a}{2}$

Thus, if the magnitudes of the two forces are equal, the resultant bisects the angle between the two forces.

4. If $a = 90^0$, i.e., if the forces are orthogonal (perpendicular) to each other, then from (1), we have

$$R^2 = P^2 + 2PQ\cos 90^0 + Q^2 = P^2 + Q^2$$

i.e. $R = (P^2+Q^2)^{\frac{1}{2}}$

Again, from (3), we have

$$\tan q = \frac{Q}{P}$$

Note : P and Q are called the component forces (or components) of the force R.

Components and resolved parts

Let the force R be given. It acts at O along OD. Let OX and OY be two given direction on different sides of OD.

Let OD make angle a with OX and b with OY.

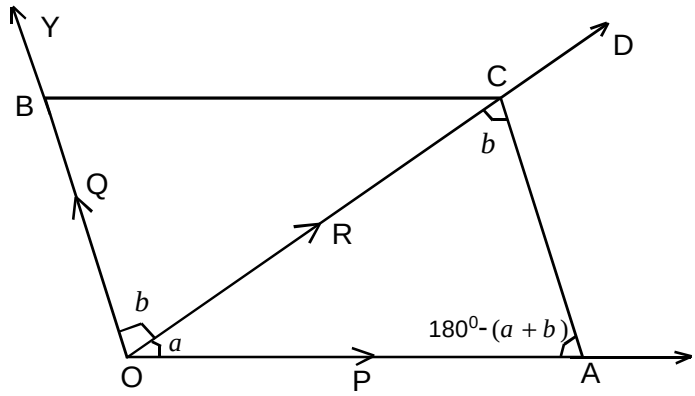


Fig 1.5

Let the force R be represented by OC completely. Through C, we draw BC parallel to OX, and AC parallel to OY respectively. We complete the parallelogram OACB. Let the force P be represented by OA completely and the force Q be represented by OB completely.

Then, P and Q are the components of R and R is the resultant of P and Q.

From the triangle OAC, we have

$$\frac{OA}{\sin\angle OCA}=\frac{AC}{\sin\angle AOC}=\frac{OC}{\sin\angle OAC}$$

or $\frac{P}{\sin b}=\frac{Q}{\sin a}=\frac{R}{\sin(180^0-(a+b))}$

or, $\frac{P}{\sin b}=\frac{Q}{\sin a}=\frac{R}{\sin(a+b)}$

$$\therefore P=\frac{R\sin b}{\sin(a+b)}$$

$$Q=\frac{R\sin a}{\sin(a+b)}$$

Thus, if R, a , b are given, we can determine P, Q uniquely

Deductions : If $a+b=90^0$ then $b=90^0-a$.

$$\therefore\ P=\frac{R\sin(90^0-a)}{\sin 90^0}=R\cos a$$

$Q = R \sin \alpha$

$R\cos a$ and $R\sin a$ are called the resolved parts or resolutes of the force R.

Corollary : If $a = 0$,

R cos a = R cos 0 = R

and R sin a = R sin0 = 0

Thus, we conclude that the resolved part of a force along its own direction is the force itself and the resolved part along its perpendicular direction is zero.

Example Resolve a force of 18N along a direction making an angle of 30^0 with the force and a second direction perpendicular to the force.

Solution

Resolved part along the 1st direction = 18 $\cos 30^0$N

$$= 18 \times \frac{\sqrt{3}}{2} N = 9\sqrt{3} N$$

Resolved part along the 2nd direction = 18 $\sin 30^0$N

$$= 18 \times \frac{1}{2} N = 9N$$

Theorem on resolved parts

The algebraic sum of the resolved parts of two forces acting at a point in any direction in their plane is equal to the resolved part of their resultant in the same direction.

Proof Let the forces P and Q act at a point O of a body. Let $\vec{OA}$ and $\vec{OB}$ represent the forces P and Q respectively. We complete the parallelogram OACB. The resultant R is represented by the diagonal $\vec{OC}$ in magnitude and direction.

Let OX and OY be two given directions perpendicular to each other in the plane of the forces.

Let a, b, q be the angles made by P, Q, R with the direction OX respectively.

We draw AL, BM, CN perpendiculars on OX, and AE perpendicular on CN.

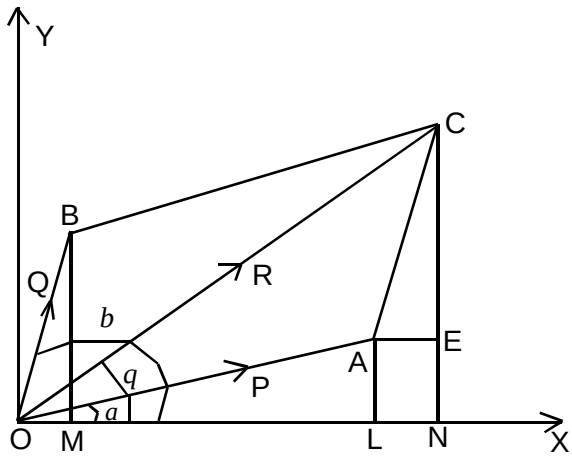


Fig 1.6

Clearly the triangles OBM and ACE are congruent

$\therefore$ OM = AE

From the right angled triangle OLA

$$\cos a = \frac{OL}{OA} \qquad \therefore OL = OA\cos a$$

From the right angled triangle ONC, we have

$$\cos q = \frac{ON}{OC} \qquad \therefore ON = OC\cos q$$

From the right angled triangle OMB, we have

$$\cos b = \frac{OM}{OB} \qquad \therefore OM = OB\cos b$$

Now, ON = OL + LN

or, ON = OL + AE

or, ON = OL + OM

or, OC $\cos q$ = OA$\cos a$ + OB$\cos b$

i.e. R$\cos q$ = P$\cos a$ + Q$\cos b$

i.e. resolved part of R along OX

= resolved part of P along OX

+ resolved part of Q along OX.

Extension: The above result can be extended to any finite number of forces. Thus, the algebraic sum of the resolved parts of any number of forces acting at a point in any direction in their plane is equal to the resolved

part of their resultant in the same direction.

Resultant of a finite number of forces acting at a point

Let P_1, P_2, P_3, ..., P_n be n number of forces acting at a point O.

We choose any two suitable perpendicular lines OX and OY in the plane of the forces.

Let the line of action of the forces P_1, P_2, P_3, ..., P_n make angles a_1, a_2, a_3, ... a_n with OX respectively.

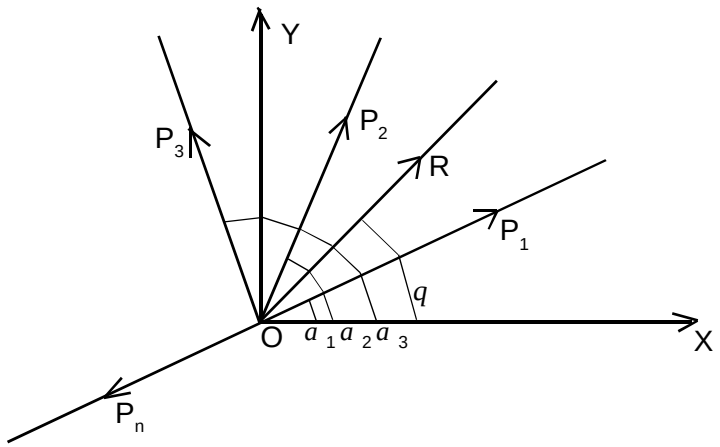


Fig 1.7

Let R be the resultant of the forces, and its line of action make angle q with OX sums of the resolved parts of the forces P_1, P_2, P_3, ..., P_n along OX and OY respectively.

Therefore,

$$R\cos q = X = P_1\cos a_1 + P_2\cos a_2 + ... + P_n\cos a_n$$

$$R\sin q = Y = P_1\sin a_1 + P_2\sin a_2 + ... + P_n\sin a_n$$

$$\therefore R^2 = X_2 + Y_2 \quad (1)$$

$$\tan q = \frac{Y}{X} \qquad (2)$$

From (1), we get the magnitude of the resultant.

From (2), we get the direction of the resultant.

Note : Here X and Y cannot be zero simultaneously. If X = 0, Y = 0, then R = 0, and in this case the system of forces will be in equilibrium.

Example 1 : Find the angle between two forces of equal magnitude acting at a point so that their resultant is also equal to either of them in magnitude.

Solution : Let the magnitude of each of the forces be P.

$\therefore$ the magnitude of the resultant is also P.

Let a be the angle between the forces.

$\therefore$ $P^2 = P^2 + P^2 + 2PP\cos a$

i.e. $1 + 2\cos a = 0$

i.e. $\cos a = -\frac{1}{2}$

$\therefore$ $a = 120^0$

$\therefore$ the angle between the forces is 120^0.

Example 2 : The line of action of the resultant of two forces of magnitude P, Q acting at point divides the angle between them in the ratio 1 : 2. Show that the magnitude of then resultant is $\frac{P^2 - Q^2}{Q}$.

Solution : Let q be the angle between the forces P and Q.

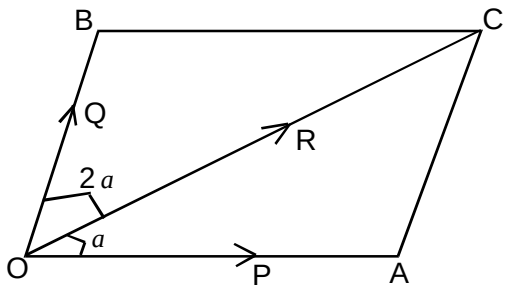


Fig 1.8

It is given that the resultant divides the angle between the forces in the ratio 1 : 2.

$\therefore$ if $\angle COA = a$, $\angle COB = 2a$

$\therefore q = a + 2a = 3a$

Now $R^2 = P^2 + Q^2 + 2PQ\cos q$

or, $R^2 = P^2 + Q^2 + 2PQ\cos^3 a$

or, $R^2 = P^2 + Q^2 + 2PQ(4\cos^3 a - 3\cos a)$

Again, $\tan a = \frac{Q\sin 3a}{P + Q\cos 3a}$

or, $(P + Q\cos 3a)\tan a = Q\sin 3a$

or, $(P + Q(4\cos^3 a - 3\cos a)\frac{\sin a}{\cos a} = Q(3\sin a - 4\sin^3 a)$

or, $P\dfrac{\sin a}{\cos a} = 2Q\sin a$

or, $\cos a = \dfrac{P}{2Q}$

$$\therefore R^2 = P^2 + Q^2 + 2PQ\left(4\left(\frac{P}{2Q}\right)^3 - \frac{3P}{2Q}\right)$$

$$= P^2 + Q^2 + \frac{P^4}{Q^2} - 3P^2$$

$$= \frac{P^2Q^2 + Q^4 + P^4 - 3P^2Q^2}{Q^2}$$

$$= \frac{P^4 + Q^4 - 2P^2Q^2}{Q^2}$$

$$= \frac{\left(P^2 - Q^2\right)^2}{Q^2}$$

$$\therefore\ R = \frac{P^2 - Q^2}{Q}$$

Example 3 : The magnitudes of the greatest and least resultants of two forces of given magnitude acting at a point are P_1 and P_2 respectively. Prove that when the forces act at an angle $2a$, their resultant is of magnitude. $\left(P_1^{\,2}\cos^2 a + P_2^{\,2}\sin^2 a\right)^{\frac{1}{2}}$

Solution : Let the forces be P and Q.

Now $P_1 = P + Q$

$P_2 = P - Q$, if we assume $P > Q$.

$\therefore\ 2P = P_1 + P_2$ i.e, $P = \dfrac{P_1 + P_2}{2}$

$2Q = P_1 - P_2$ i.e, $Q = \dfrac{P_1 - P_2}{2}$

Let the resultant of P and Q be R.

$$\therefore\ R^2 = P^2 + Q^2 + 2PQ\cos 2a$$

$$= \frac{1}{4}\left(P_1 + P_2\right)^2 + \frac{1}{4}\left(P_1 - P_2\right)^2 + 2.\frac{P_1 + P_2}{2}.\frac{P_1 - P_2}{2}.\cos 2\alpha$$

$$= \frac{1}{4}\left[2P_1^2 + 2P_2^2 + 2(P_1^2 - P_2^2)\cos 2a\right]$$

$$= \frac{1}{4}\left[2P_1^2(1+\cos 2a) + 2P_2^2(1-\cos 2a\right]$$

$$= \frac{1}{4}\left[2P_1^2 . 2\cos^2 a + 2P_2^2 . 2\sin^2 a\right]$$

$$= P_1^2\cos^2 a + P_2^2\sin^2 a$$

$$\therefore R = \left(P_1^2\cos^2 a + P_2^2\sin^2 a\right)^{\frac{1}{2}}$$

Example 4 : The magnitude of the resultant of two forces P and Q acting at an angle q is equal to (2n+1)$\sqrt{P^2+Q^2}$. When they act at an angle $(90^0 - q)$, the magnitude of the resultant is (2n–1)$\sqrt{P^2+Q^2}$. Show that

$$\tan q = \frac{n-1}{n+1}$$

Solution : We have

$$(2n+1)^2\left(\sqrt{P^2+Q^2}\right)^2 = P^2 + Q^2 + 2PQ\cos q$$

i.e. $(4n^2+4n+1)(P^2+Q^2) - (P^2+Q^2) = 2PQ\cos q$

i.e. $(4n^2+4n+1-1)(P^2+Q^2) = 2PQ\cos q$

i.e. $4n + (n+1)(P^2+Q^2) = 2PQ\cos q$ (1)

Again, $(2n-1)^2\left(\sqrt{P^2+Q^2}\right)^2 = P^2 + Q^2 + 2PQ\cos(90^0 - q)$

i.e. $(4n^2-4n+1)(P^2+Q^2) - (P^2+Q^2) = 2PQ\sin q$

i.e. $(4n^2-4n+1-1)(P^2+Q^2) = 2PQ\sin q$

i.e. $4n(n-1)(P^2+Q^2) = 2PQ\sin q$ (2)

From (1) and (2),

$$\frac{2PQ\sin q}{2PQ\cos q} = \frac{4n(n-1)(P^2+Q^2)}{4n(n+1)(P^2+Q^2)}$$

i.e. $\tan q = \frac{n-1}{n+1}$

Example 5 : Two forces of magnitude P and 2P act on a particle. If the angle between the forces is 120^0, find the angle made by the resultant with the greater force.

Solution : Let q be the angle made by the resultant with the force 2P (the greater force).

$$\therefore \tan q = \frac{P\sin 120^0}{2P + P\cos 120^0}$$

$$= \frac{\frac{\sqrt{3}P}{2}}{2P - \frac{P}{2}} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\therefore q = 30^0$$

$\therefore$ the resultant makes an angle of 30^0 with the greater force.



CHECK YOUR PROGRESS

Q 1: Two forces of magnitude 2N and 3N act at a point at an angle 60^0 to each other. find the magnitude of the resultant and the angle made by the resultant with the direction of the force 2N.

Q 2: Two forces of equal magnitudes 4N and 4N act at a point at an angle 120^0 to each other. Find the resultant and the angle made by the resultant with either of the forces.

Q 3: Resultant of two forces acting at a point at right angle is 8N. If the resultant makes an angle of 60^0 with one of the forces, find the forces.

Q 4: The resultant of two forces of magnitudes P and 2P acting at a point is $P\sqrt{3}$. Find the angle between the forces and the angle made by the resultant with the force of P.

Q 5: Two forces of equal magnitude act on a particle. Find the angle between them when the square of their resultant is equal to three times their product.

Q 6: Two forces act at a point and are such that if the direction of one is reversed, the direction of the resultant is turned through a right angle. Prove that the two forces are equal in magnitude.

Q 7: The resultant of two forces P and Q acting at a point is equal to Q in magnitude and that of P and 2Q (acting in the same

direction) is also Q. Show that $P = \sqrt{3}Q$ and the angle between the forces is 150^0.

Q 8: The resultant of two forces P and Q is $\sqrt{3}Q$ and the resultant is at an angle of 30^0 with P. Show that either P = Q or P = 2Q.

Q 9: The resultant of two forces P and Q acting at a point is R; if Q is doubled, R is doubled. If Q is reversed, R is again doubled. Show that

$P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$



1.6 LET US SUM UP

- A force is that which changes or tends to change the state of rest of a body or of its uniform motion in a straight line.
- A force has three characteristics, viz, (i) a magnitude, (ii) a direction, (iii) a point of application.
- A force has both magnitude and direction. So, it is a vector quantity.
- There are different systems of units for measuring physical quantities, viz, (i) CGS system, (ii) MKS system, (iii) FPS system.
  MKS system is nothing but an extension of CGS system. It is used throughout the world nowadays.
- There are different types of forces, viz, attraction and repulsion; tension and thru; action and reaction; friction.
- If two forces acting at a point are represented in magnitude and direction by the sides of a parallelogram drawn from that point, their resultant is represented in magnitude and direction by the diagonal of the parallelogram drawn from that point.
  This is the fundamental theorem of statics.
- The magnitude R of two forces P and Q acting at a point at an angle a is given by $R^2 = P^2 + 2PQ\cos a + Q^2$.
  The direction of the resultant R is given by the relations

$R\cos q = P + Q\cos a$

$R\sin q = Q\sin a$

- The algebraic sum of the resolved parts of two forces acting at a point in any direction in their plane is equal to the resolved part of their resultant in the same direction.



1.7 FURTHER READINGS

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridgee University Press.
2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and Ashok Gupta.
3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and Sankara subramanian. Vikas Publishing House Pvt. Ltd.



1.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: We have

$$R^2 = 2^2 + 2\times 2\times 3\cos 60^0 + 3^2$$

$$= 4 + 2\times 2\times 3\times \frac{1}{2} + 9 = 19$$

$$\therefore\ R = \sqrt{19}\ N$$

Also $\tan q = \dfrac{3\sin 60^0}{2 + 3\cos 60^0}$ (Here P = 2, Q = 3)

$$= \frac{\frac{3\sqrt{3}}{2}}{2 + 3\times\frac{1}{2}} = \frac{3\sqrt{3}}{7}$$

$$\therefore q = \tan^{-1}\frac{3\sqrt{3}}{7}$$

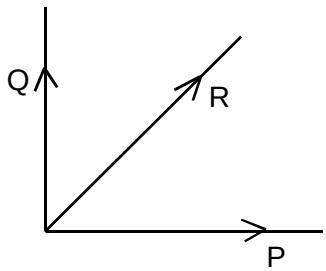
Ans to Q No 2: We know that if the forces are of equal magnitude P, then

$$R = 2p\cos\frac{a}{2}, \quad \text{and } q = \frac{a}{2}$$

$$\therefore\ R = 2\times 4\times\cos\frac{120^0}{2}N$$

$$= 2\times 4\times\cos 60^0 N = 2\times 4\times\frac{1}{2}N = 4N$$

$$q = \frac{120^0}{2} = 60^0$$



Ans to Q No 3: We have $R^2 = P^2 + Q^2$

and $\tan q = \frac{Q}{P}$

Now, $8^2 = P^2 + Q^2$ i.e. $P^2 + Q^2 = 64$

$\tan 60^0 = \frac{Q}{P}$ i.e. $\sqrt{3} = \frac{Q}{P}$ i.e. $Q = \sqrt{3}P$

$\therefore\ P^2 + 3P^2 = 64$ or $4P^2 = 64$ or $P^2 = 16$

$\therefore\ P = 4$ N

and $Q = 4\sqrt{3}N$

Ans to Q No 4: Let a be the angle between the forces.

$$\therefore\ \left(P\sqrt{3}\right)^2 = P^2 + 2\times P\times 2P\cos a + (2P)^2$$

or, $3P^2 = P^2 + 4P^2\cos a + 4P^2$

or, $2\cos a = -1$

or, $\cos a = -\frac{1}{2}$

$$\therefore a = 120^0$$

Let q be the angle made by the resultant with the force P.

$$\therefore\ P\sqrt{3}\sin q = 2P\sin 120^0 \ (R\sin q = Q\sin a)$$

or, $\sqrt{3}\sin q = 2\times\frac{\sqrt{3}}{2}$

or, $\sin q = 1$

$$\therefore q = 90^0$$

Ans to Q No 5: Let R be the resultant and a be the angle between the forces.

Let the magnitude of each force be P.

$\therefore R^2 = P^2 + 2PP\cos a + P^2$

$= 2P^2 + 2P^2\cos a = 2P^2(1+\cos a)$

Also $R^2 = 3PP = 3P^2$

$\therefore 3P^2 = 2P^2(1+\cos a)$

or $1+\cos a = \frac{3}{2}$

or $\cos a = \frac{1}{2}$

$\therefore a = 60^0$

$\therefore$ the angle between the forces is 60^0.

Ans to Q No 6: Let two forces P and Q act at a point O. Let a be the angle between the forces.

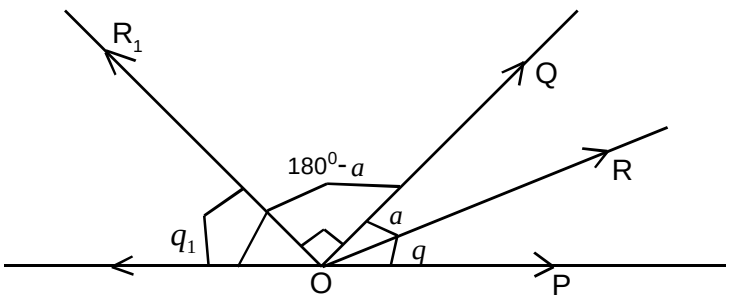


Fig 1.9

Let q be the angle made by the resultant with the force P.

$\therefore \tan q = \frac{Q\sin a}{P + Q\cos a}$

Let the force P be reversed. Then the resultant is turned through a right angle (given). Let the new resultant make an angle q_1, with P (when reversed).

Now, $q + 90^0 + q_1 = 180^0$

$\therefore q_1 = 180^0 - 90^0 - q = 90^0 - q$

Now $\tan q_1 = \frac{Q\sin(180^0 - a)}{P + Q\cos(180^0 - a)}$

or, $\tan(90^0 - q) = \dfrac{Q\sin a}{P - Q\cos a}$

or, $\cot q = \dfrac{Q\sin a}{P - Q\cos a}$ (2)

From (1) and (2)

or, $\dfrac{Q\sin a}{P + Q\cos a} \cdot \dfrac{Q\sin a}{P - Q\cos a} = \tan q \ . \cot q$

or, $\dfrac{Q^2\sin^2 a}{P^2 - Q^2\cos^2 a} = 1$

or, $P^2 - Q^2\cos^2 a = Q^2\sin^2 a$

or, $P^2 = Q^2(\sin^2 a + \cos^2 a) = Q^2$

$\therefore$ P = Q

Ans to Q No 7: Let a be the angle between the forces.

$\therefore Q^2 = P^2 + 2PQ\cos a + Q^2$

or, $P^2 + 2PQ\cos a = 0$

or, $P(P + 2Q\cos a) = 0$

or, $P + 2Q\cos a = 0$

or, $2Q\cos a = -P$ (1)

Again, $Q^2 = P^2 + 4PQ\cos a + 4Q^2$

or, $P^2 + 4PQ\cos a + 3Q^2 = 0$

or, $P^2 + 2P(2Q\cos a) + 3Q^2 = 0$

or, $P^2 + 2P(-P) + 3Q^2 = 0$ from (1)

or, $P^2 = 3Q^2$

$\therefore P = \sqrt{3}\,Q$

$\therefore$ from (1) $2Q\cos a = -\sqrt{3}\,Q$

$\therefore \cos a = -\dfrac{\sqrt{3}}{2}$

or, $\cos a = -\cos 30^0$

or, $\cos a = -\cos(180^0 - 30^0)$

$\therefore a = 150^0$

i.e. the angle between the forces is 150^0.

Ans to Q No 8: Let a be the angle between the force.

$\therefore \left(\sqrt{3}Q\right)^2 = P^2 + 2PQ\cos a + Q^2$

or, $3Q^2 = P^2 + 2Q\cos a + Q^2$

or, $P^2 + 2PQ\cos a - 2Q^2 = 0$ (1)

Again $\sqrt{3}Q\cos 30^0 = P + Q\cos a$ $(R\cos q = P + Q\cos a)$

or, $\sqrt{3}Q \,.\, \frac{\sqrt{3}}{2} = P + Q\cos a$

or, $3Q = 2P + 2Q\cos a$

or, $2Q\cos a = 3Q - 2P$ (2)

From (1) and (2),

$P^2 + P(3Q - 2P) - 2Q^2 = 0$

or, $P^2 + 3P\,Q - 2P^2 - 2Q^2 = 0$

or, $P^2 - 3PQ + 2Q^2 = 0$

or $(P - Q)(P - 2Q) = 0$

$\therefore P = Q$ or $P = 2Q$

Ans to Q No 9: Let a be the angle between P and Q.

$\therefore R^2 = P^2 + 2PQ\cos a + Q^2$ (1)

Also, $(2R)^2 = P^2 + 2P(2Q\cos a) + (2Q)^2$

or, $4R^2 = P^2 + 4PQ\cos a + 4Q^2$ (2)

Again, $(2R)^2 = P^2 + 2PQ\cos(180^0 - a) + Q^2$

or, $4R^2 = P^2 - 2PQ\cos a + Q^2$ (3)

Multiplying (1) by 2, we get

$2R^2 = 2P^2 + 4PQ^2\cos a + 2Q^2$ (4)

Subtracting (4) from (2), we get

$2R^2 = -P^2 + 2Q^2$

or $P^2 - 2Q^2 + 2R^2 = 0$ (5)

Again, adding (1) and (3), we get

$5R^2 = 2P^2 + 2Q^2$

or, $2P^2 + 2Q^2 - 5R^2 = 0$ (6)

From (5) and (6), we get

$$\frac{P^2}{10-4} = \frac{Q^2}{-(-5-4)} = \frac{R^2}{4+2}$$

or, $\frac{P^2}{6} = \frac{Q^2}{9} = \frac{R^2}{6}$

or, $\frac{P^2}{2} = \frac{Q^2}{3} = \frac{R^2}{2}$

$\therefore \frac{P}{\sqrt{2}} = \frac{Q}{\sqrt{3}} = \frac{R}{\sqrt{2}}$



1.9 MODEL QUESTIONS

Q 1: Two forces P and Q act at a point at an angle q and their resultant R makes an angle a with P. If P = 24, R = 36, a = 90^0, find Q.

Q 2: Two unequal forces act at a point. Show that their resultant is nearer the greater force.

Q 3: The magnitude of the resultant of two forces each of magnitude P acting at a point is R. If $R^2 = 3P^2$. Show that the forces act at an angle of 60^0.

Q 4: The resultant of two forces P and Q acting at a point at an angle q is R. If the first force is doubled in magnitude. Show that the magnitude of the new resultant is $\sqrt{2P^2 + 2R^2 - Q^2}$.

Q 5: The resultant of forces P and Q acting at a point at angle 90^0, q and $90^0 - q$ are of magnitude R, nR and (n + 2) R respectively. Show that $\frac{n+3}{n-1}$.

Q 6: The resultant of two forces P and Q acting at a point is of magnitude P. If the magnitude of the first force is doubled, show that the new resultant is of magnitude $\sqrt{4P^2 - Q^2}$.

Q 7: The resultant of two forces P and Q acting at a point is R. If the direction of the second force is reversed and its magnitude changes

to $(R^2 - P^2)/Q$, Show that the magnitude of the resultant is not altered.

Q 8: If the resultant of two forces acting on a particle be at right angle to one of them and its magnitude be one-third the magnitude of the other, Show that the magnitude of the larger force to that of the smaller force is $3:2\sqrt{2}$.

Q 9: R is the resultant of the forces P and Q acting at the point. If a transversal cuts the lines of action of P, Q, R at the points L, M, N respectively. Show that $\frac{P}{OL}+\frac{Q}{OM}=\frac{R}{ON}$.

Q 10: The magnitude of the resultant of two forces of equal magnitude acting at angle $2q$ is twice that when they act at angle $2f$. Show that $\cos q = 2\cos f$.

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UNIT 2: COPLANAR FORCES

UNIT STRUCTURE

2.1 Learning Objectives

2.2 Introduction

2.3 Lami's Theorem

2.4 Equilibrium of a particle under several coplanar forces

2.5 Parallel Forces

2.6 Let Us Sum Up

2.7 Further Readings

2.8 Answers To Check Your Progress

2.9 Model Questions

2.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- Lami's Theorem
- equilibrium of a particle under several coplanar forces
- parallel forces

2.2 INTRODUCTION

If two forces act on equal masses under similar conditions and produce equal accelerations, the forces are said to be equal. If a number of forces act on a body, the body may move or may be at rest. If the body remains at rest, the body is said to be in equilibrium, and the forces are also said to be in equilibrium. In this unit, we will study Lami's theorem, equilibrium of a particle under several coplanar forces and parallel forces.

2.3 LAMI'S THEOREM

If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two forces.

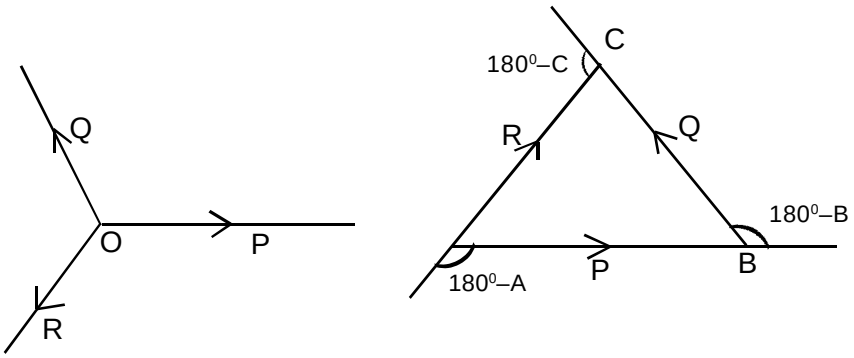


Fig 2.1

Let three forces P, Q, R acting on O be at equilibrium.

It is required to prove that $\frac{P}{\sin(\widehat{Q,R})} = \frac{Q}{\sin(\widehat{R,P})} = \frac{R}{\sin(\widehat{P,Q})}$

Where $(\widehat{Q, R})$ means angle between Q and R,

$(\widehat{R, P})$ means angle between R and P,

$(\widehat{P, Q})$ means angle between P and Q.

Let ABC be a triangle whose sides AB, BC, CA are parallel to the directions of P, Q, R respectively.

$\therefore$ the lengths AB, BC, CA are proportional to the magnitudes of P, Q, R respectively.

$$\therefore \frac{AB}{P} = \frac{BC}{Q} = \frac{CA}{R} \qquad (1)$$

Again from trigonometry, we know that

$$\therefore \frac{AB}{Sin\,C} = \frac{BC}{Sin\,A} = \frac{CA}{Sin\,B} \qquad (2)$$

From (1) and (2),

$$\frac{P}{Sin\,C} = \frac{Q}{Sin\,A} = \frac{R}{Sin\,B}$$

i.e. $$\frac{P}{Sin(180^0 - C)} = \frac{Q}{Sin(180^0 - A)} = \frac{R}{Sin(180^0 - B)}$$

Now, $180^0 - C$ = angle between Q and R

$180^0 - A$ = angle between R and P

$180^0 - B$ = angle between P and Q

$$\therefore \quad \frac{P}{Sin(\widehat{Q\,P})} = \frac{Q}{Sin(\widehat{R,\,P})} = \frac{R}{Sin(\widehat{P\,Q})}$$

2.4 EQUILIBRIUM OF A PARTICLE UNDER SEVERAL COPLANAR FORCES

We consider the following three cases.

Case I : Two forces are acting on a particle. The particle will be in equilibrium, i.e., the forces are in equilibrium if

(i) they have equal magnitude,

(ii) they act along the same line,

and (iii) they act in opposite directions.

Case II : Three non-colinear forces are acting on a particle. The particle will be in equilibrium, i.e., the forces are in equilibrium if

(i) the forces can be represented by the sides of a triangle taken in order

(ii) each force is proportional to the sine of the angle between the other two forces (known as Lami's Theorem)

or (iii) if the sum of their resolved along each of two perpendicular directions in their plane is zero.

Case III : A finite number of forces, no two of which are colinear, are acting on a particle. The particle will be in equilibrium, i.e., the forces are in equilibrium if

(i) the forces can be represented by the sides of a polygon taken in order.

or (ii) the forces are coplanar and the sum of their resolved parts along each of two perpendicular directions in their plane is zero.

or (iii) the point at which the given forces act is the centroid of the extremities of the line segments representing the forces.

Example 1 : Examine whether three forces whose magnitudes are in the proportion 5 : 3 : 2, acting on a particle, can be kept in equlibrium under any circumstance.

Solution : Let the magnitudes of the forces be 5k, 3k, 2k.

Equilibrium is possible when the force of magnitude 5k acts in one

direction and the forces of magnitudes 3k and 2k act in the opposite direction along the same straight line.



CHECK YOUR PROGRESS 1

Q 1: Examine whether three forces whose magnitudes are in the proportion 3 : 2 : 1 acting on a particle, can be kept in equilibrium under any circumstance.

Q 2: Examine whether three forces whose magnitudes are in the proportion 3 : 5 : 10 acting on a particle can be kept in equilibrium.

Q 3: Three equal forces acting at a point are equally inclined to each other. Show that they are in equilibrium.

Q 4: Three forces P, Q, R act at the in centre I of the triangle ABC in the direction IA, IB, IC respectively. If the forces are in equilibrium, prove that $P^2 : Q^2 : R^2 = a(b + c – a) : b(c + a –b) : c(a + b + c)$.

Q 5: Three forces P, Q, R act at the centroid G of the triangle ABC in the directions GA, GB, GC respectively. If the forces are in equilibrium, prove that

P : Q R = GA : GB : GC

Q 6: Three forces P, Q, R act at the circumcentre O of the triangle ABC in the directions OA, OB, OC respectively. If the forces are in equilibrium, prove that P : Q : R.

$$= \frac{1}{b^2}+\frac{1}{c^2}-\frac{a^2}{b^2c^2} : \frac{1}{c^2}+\frac{1}{a^2}-\frac{b^2}{c^2a^2} : \frac{1}{a^2}+\frac{1}{b^2}-\frac{c^2}{a^2b^2}$$

Q 7: Three non-collinear forces acting at a point O are in equilibrium. A circle is drawn through O to cut the lines of action of the forces in A, B, C respectively. Prove that the forces are proportional to the sides of the triangle ABC.

Q 8: Forces acting at a point are represented in magnitude and direction by 2AB, 3BC, 2CD, DA, CA and DB where ABCD is a quadrilateral. Show that the forces are in equilibrium.

Q 9: Forces acting at a point are represented in magnitude and direction by AB, 2BC, 2CD, 2DE, 2EF, FA and FB where ABCDEF is a hexagon. Show that the forces are in equilibrium.

Q 10: ABCDE is a pentagon. Show that the forces acting at a point and represented in magnitude and direction by AB, 3BC, 2CA, CD, DE, 2EB and AE are in equilibrium.

2.5 PARALLEL FORCES

Two forces are said to be parallel if their lines of action are parallel. They are said to be like parallel if they act in the same direction, whereas they are called unlike parallel if they act in opposite directions.

Clearly the forces are not concurrent. They are assumed to act on a rigid body.

Resultant of two like parallel forces

Let P, Q be two like parallel forces acting at the points A, B respectively of a rigid body.

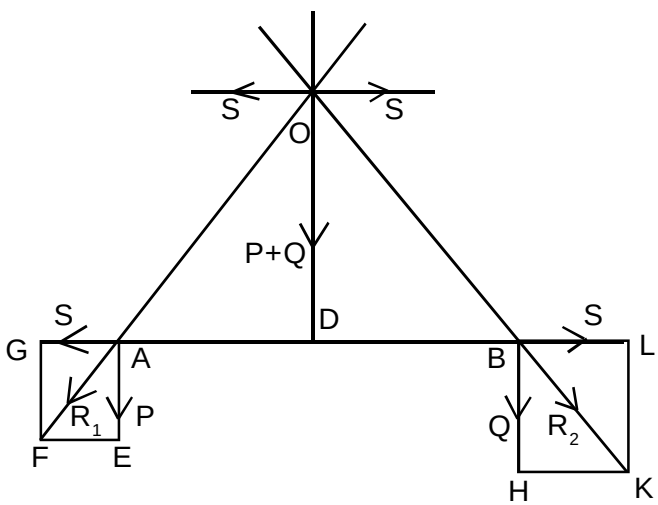


Fig 2.2

Let P, Q be represented by AE, BH respectively.

We join AB and introduce two equal and opposite force S, S at A, B represented by AG, BL respectively.

Since the forces balance each other, they will have no effect on the system.

We complete the parallelograms AEFG and BHKL.

The resultant of P and S is R_1 along AF, and that of Q and S is R_2 along BK.

We produce FA and KB to meet at O.

We draw OD parallel to AE (or BH) to meet AB at D.

The resultants R_1 and R_2 may be supposed to act at O. Thus, R_1 at O is equivalent to P at O along OD and S at O parallel to AG. Also R_2 at O is equivalent to Q at O along OD and S at O parallel to BL.

The two forces S, S at O balance each other. So, we are left with P + Q along OD, which is the resultant of the like parallel forces P, and Q.

Now, we find the position of D where the line of action of P + Q meets AB.

Clearly, the triangles AFE and OAD are similar.

$$\therefore \frac{AE}{OD} = \frac{EF}{DA}$$

or, $\frac{P}{OD} = \frac{S}{DA}$ i.e., $\frac{P}{S} = \frac{OD}{DA}$ (1)

Also the triangles BHK and ODB are similar.

$$\therefore \frac{BH}{OD} = \frac{HK}{DB}$$

or, $\frac{Q}{OD} = \frac{S}{DB}$ i.e., $\frac{Q}{S} = \frac{OD}{DB}$ (2)

From (1) and (2), $\frac{P}{S}.\frac{S}{Q} = \frac{OD}{DA}.\frac{DB}{OD}$

i.e. $\frac{P}{Q} = \frac{DB}{DA}$

Thus, D divides AB internally in the inverse ratio of the forces.

Note : If P = Q, the resultant is 2P and in this case, D is the middle point of AB.

Resultant of two unequal, unlike parallel forces

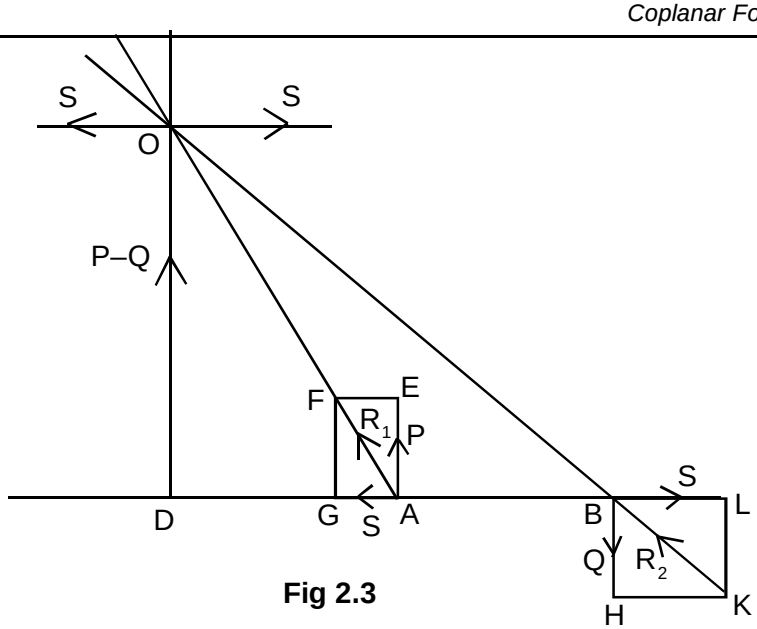


Fig 2.3

Let P, Q be two unlike parallel forces acting at the points A, B respectively of a right body, where P > Q.

Let P, Q be represented by AE, BH respectively.

We join AB and introduce two equal and opposite forces S, S at A, B represented by AG, BL respectively.

Since the forces balance each other they will have no effect on the system.

We complete the parallelograms AEFG and BHKL.

The resultant of P and S is R_1 along AF, and that of Q and S is R_2 along KB.

We produce AF and KB to meet at O.

We draw OD parallel to AE (or BH) to meet BA produced at D.

The resultant R_1 and R_2 may be supposed to act at O. Thus, R_1 at O is equivalent to P at O along DO, and S at O parallel to AG. Also R_2 at O is equivalent to Q at O along OD and parallel to BL.

The two forces S, S at O balance each other. So, we are left with P– Q along DO, which is the resultant the unlike parallel forces P and Q.

Now we find the position of D where the line of action of P–Q meets BA produced.

Clearly, the triangles AFG, and AOD are similar.

$$\therefore \frac{GF}{DO} = \frac{AG}{AD}$$

or, $\frac{GF}{AG}=\frac{DO}{AD}$ i.e., $\frac{P}{S}=\frac{DO}{AD}$ (1)

Also, the triangles BHK and ODB are similar.

$$\therefore \frac{BH}{OD}=\frac{HK}{DB}$$

or, $\frac{BH}{HK}=\frac{OD}{DB}$ i.e., $\frac{Q}{S}=\frac{OD}{DB}$ (2)

From (1) and (2), $\frac{P}{S}\cdot\frac{Q}{S}=\frac{DO}{AD}\cdot\frac{BD}{OD}$

i.e., $\frac{P}{Q}=\frac{DB}{DA}$

Thus, D divides BA externally in the inverse ratio of the forces.

Example 2 : Two like parallel forces 11 newton and 19 newton are acting at A and B of a rigid body. If AB = 10m, find the magnitude and the point of application of the resultant.

Solution : Let P = 11 newton

Q = 19 newton

∴ R = (P + Q) newton

= (11 + 19) newton

= 30 newton.

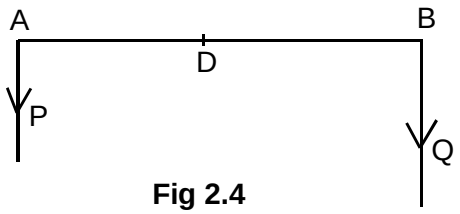


Fig 2.4

Let D be the point of application of the resultant.

∴ P. DA = Q . DB

or, 11. DA = 19. (10 – DA)

or, 11. DA = 190 – 19DA

or, 30DA = 190

$$\therefore DA=\frac{19}{3}m$$



CHECK YOUR PROGRESS 2

Q 1: Two like parallel forces P and Q acting at A and B have a resultant R acting at D on AB. If P = 6 newton, AD = 9m and DB = 8m, find Q and R.

Q 2: The position of the resultant of two like parallel forces remains unaltered, when the positions of the forces are interchanged. Show that the forces are equal in magnitude.

Q 3: Two unlike parallel forces P and Q acting at A and B have a resultant R acting at D on AB produced. If P = 5 newton, AD = 12m, AB = 9m, find Q and R.

Q 4: P and Q are two unlike parallel forces where P > Q. If the magnitude of each of the force is increased by the same amount, prove that the line of action of the resultant will move further off from P.

Q 5: P and Q are two like parallel forces. If P is moved parallel to itself through a distance x, prove that the resultant moves through a distance.

$$\frac{Px}{P+Q}.$$

Q 6: Three like parallel forces P, Q, R are acting at the angular points A, B, C of the triangle ABC. If the line of action of the resultant passes through

(i) the centroid, prove that P = Q = R,

(ii) the circumcentre, prove that

$$\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

(iii) the incentre, prove that

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}$$

(iv) the orthocentre, prove that

$$\frac{P}{\tan A} = \frac{Q}{\tan B} = \frac{R}{\tan C}$$



2.9 LET US SUM UP

- If a number of forces act on a body, the body may move or may be at rest. If the body remains at rest, the body is said to be in equilibrium, and the forces are also said to be in equilibrium.
- If three forces acting on a particle are in equilibrium, each force is proportional to the sine of the angle between the other two forces. (This is konwn as Lami's Theorem)
- Two forces are said to be parallel if their lines of action are parallel. They are said to be like parallel if they act in the same direction, whereas they are called unlike parallel if they act in opposite directions. They are assumed to act on a rigid body.



2.10 FURTHER READING

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridge University Press.
2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and Ashok Gupta.
3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and Sankara subramanian. Vikas Publishing House Pvt. Ltd.



2.11 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS 1

Ans to Q No 1: Let the magnitudes of the forces be 3k, 2k, k.

Equilibrium is possible when the force of magnitude 3k acts in one direction and the forces of magnitudes 2k and k act in the opposite direction along the same straight line.

Ans to Q No 2: Let the magnitudes of the forces be 3k, 5k, 10k.

Equilibrium is not possible as the forces cannot be represented by the sides of a triangle. It is not possible to draw a triangle with sides 3k, 5k, 10k units, as 3k + 5k < 10k.

Ans to Q No 3: We are given that three equal forces acting at a point are equally inclined to each other. Therefore, the forces can be represented by the sides of a triangle taken in order. Therefore, the forces are in equilibrium.

Ans to Q No 4: Given that the forces are in equilibrium.

$\therefore$ by Lami's theorem,

$$\frac{P}{\sin\angle BIC}=\frac{Q}{\sin\angle CIA}=\frac{R}{\sin\angle AIB}$$

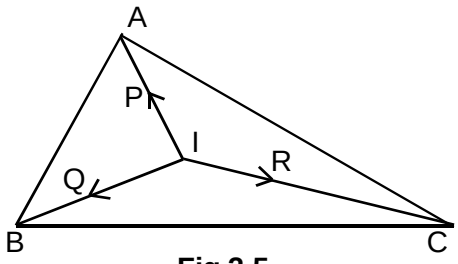


Fig 2.5

Now $\angle BIC+\frac{1}{2}\angle B+\frac{1}{2}\angle C=180^0$

$$\therefore \angle BIC=180^0-\frac{1}{2}(\angle B+\angle C)$$

$$=180^0-\frac{1}{2}(180^0-\angle A) \quad \left[\mathbf{Q}\ \angle A+\angle B+\angle C=180^0\right]$$

$$=90^0+\frac{A}{2}$$

Similarly $\angle CIA=90^0+\frac{B}{2}$, $\angle AIB=90^0+\frac{C}{2}$

$$\therefore \frac{P}{\sin\left(90^0+\frac{A}{2}\right)}=\frac{Q}{\sin\left(90^0+\frac{B}{2}\right)}=\frac{R}{\sin\left(90^0+\frac{C}{2}\right)}$$

$$\Rightarrow \frac{P}{\cos\frac{A}{2}}=\frac{Q}{\cos\frac{B}{2}}=\frac{R}{\cos\frac{C}{2}}$$

$$\Rightarrow \frac{P^2}{\cos^2\frac{A}{2}}=\frac{Q^2}{\cos^2\frac{B}{2}}=\frac{R^2}{\cos^2\frac{C}{2}}$$

Now, $\cos A = 2\cos^2\frac{A}{2} - 1$

$$\therefore 2\cos^2\frac{A}{2} = 1 + \cos A$$

$$= 1 + \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(b+c+a)\,(b+c-a)}{2bc}$$

Simarly,

$$\cos^2\frac{B}{2} = \frac{(c+a+b)(c+a-b)}{2ca}$$

$$\cos^2\frac{C}{2} = \frac{(a+b+c)(a+b-c)}{2ab}$$

$$\therefore\ P^2 : Q^2 : R^2$$

$$= \frac{(b+c+a)(b+c-a)}{4bc} : \frac{(c+a+b)(c+a-b)}{4ca} : \frac{(a+b+c)(a+b-c)}{4ab}$$

$$= a(b+c-a) : b(c+a-b) : c(a+b-c)$$

Ans to Q No 5: Given that the forces are in equilibrium.

$\therefore$ by Lami's theorem

$$\frac{P}{\sin\angle BGC} = \frac{Q}{\sin\angle CGA} = \frac{R}{\sin\angle AGB}$$

We know that area of the triangle GBC

= area of the triangle GCA

= area of the triangle GAB

$$\therefore \frac{1}{2}GB.GC\sin\angle BGC = \frac{1}{2}GC.GA\sin\angle CGA$$

$$= \frac{1}{2}GA.GB\sin\angle AGB$$

$$\Rightarrow \frac{\sin\angle BGC}{GA} = \frac{\sin\angle CGA}{GB} = \frac{\sin\angle AGB}{GC} \quad (2)$$

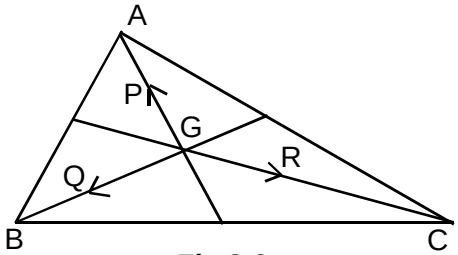


Fig 2.6

From (1) and (2)

$$\frac{P}{GA}=\frac{Q}{GB}=\frac{R}{GC}$$

Ans to Q No 6: Given that the forces are in equilibrium.

$\therefore$ by Lami's theorem

$$\frac{P}{\sin\angle BOC}=\frac{Q}{\sin\angle COA}=\frac{R}{\sin\angle AOB}$$

$$\Rightarrow \frac{P}{\sin 2A}=\frac{Q}{\sin 2B}=\frac{R}{\sin 2C}$$

$$\Rightarrow \frac{P}{2\sin A\cos A}=\frac{Q}{2\sin B\cos B}=\frac{R}{2\sin C\cos C}$$

$$\Rightarrow \frac{P}{\sin A\cos A}=\frac{Q}{\sin B\cos B}=\frac{R}{\sin C\cos C} \quad (1)$$

Again, we know that

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k, \text{ say}$$

$$\therefore \sin A=ka,\ \sin B=kb,\ \sin C=kc$$

$\therefore$ from (1),

$$\frac{P}{ka\cos A}=\frac{Q}{kb\cos B}=\frac{R}{kc\cos C}$$

i.e. $$\frac{P}{a\cos A}=\frac{Q}{b\cos B}=\frac{R}{c\cos C} \qquad (2)$$

Now $$a\cos A=a\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{abc}{2}\ \frac{b^2+c^2-a^2}{b^2c^2}$$

$$=\frac{abc}{2}\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{a^2}{b^2c^2}\right)$$

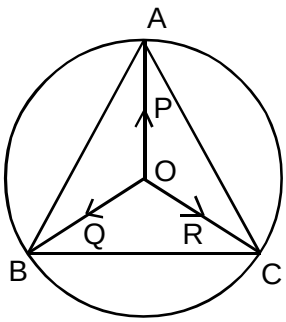


Fig 2.7

Similarly,

$$b\cos B=\frac{abc}{2}\left(\frac{1}{c^2}+\frac{1}{a^2}-\frac{b^2}{c^2a^2}\right)$$

$$c\cos C=\frac{abc}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{c^2}{a^2b^2}\right)$$

$\therefore$ from (2)

P : Q : R

$$= \frac{abc}{2}\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{a^2}{b^2c^2}\right):\frac{abc}{2}\left(\frac{1}{c^2}+\frac{1}{a^2}-\frac{b^2}{c^2a^2}\right):\frac{abc}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{c^2}{a^2b^2}\right)$$

$$= \left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{a^2}{b^2c^2}\right):\left(\frac{1}{c^2}+\frac{1}{a^2}-\frac{b^2}{c^2a^2}\right):\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{c^2}{a^2b^2}\right)$$

Ans to Q No 7: Given that the forces are in equilibrium.

$\therefore$ by Lami's theorem

$$\frac{P}{\sin\angle BOD}=\frac{Q}{\sin\angle DOA}=\frac{R}{\sin\angle AOB}$$

Now $\angle BOD+\angle BOC=180^0$

$\angle DOA+\angle AOC=180^0$

$\angle AOB+\angle BOA=180^0$

$\therefore \sin\angle BOD=\sin\left(180^0-\angle BOC\right)$

$=\sin\angle BOC$

$=\sin\angle BAC$

$\sin\angle DOA=\sin\left(180^0-\angle AOC\right)$

$=\sin\angle AOC$

$=\sin\angle ABC$

$\sin\angle AOB=\sin\left(180^0-\angle BCA\right)$

$=\sin\angle BCA$

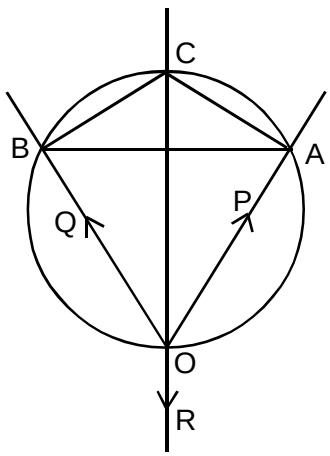


Fig 2.8

$\therefore$ from (1)

$$\frac{P}{\sin\angle BAC}=\frac{Q}{\sin\angle ABC}=\frac{R}{\sin\angle BCA} \qquad (2)$$

Again, from the triangle ACB

$$\frac{\sin\angle BAC}{BC}=\frac{\sin\angle ABC}{CA}=\frac{\sin\angle BCA}{AB} \qquad (3)$$

$\therefore$ from (2) and (3)

$$\frac{P}{BC}=\frac{Q}{CA}=\frac{R}{AB}$$

8 and 9. Left as exercise

Ans to Q No 10: We have

$$\overrightarrow{AB}+3\overrightarrow{BC}+2\overrightarrow{CA}+\overrightarrow{CD}+\overrightarrow{DE}+2\overrightarrow{EB}+\overrightarrow{AE}$$

$$=\left(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}\right)+2\overrightarrow{BC}+2\overrightarrow{CA}+2\overrightarrow{EB}+\overrightarrow{AE}$$

$= \overrightarrow{AE} + 2\overrightarrow{BC} + 2\overrightarrow{CA} + 2\overrightarrow{EB} + \overrightarrow{AE}$

$= 2\overrightarrow{AE} + 2\overrightarrow{BC} + 2\overrightarrow{CA} + 2\overrightarrow{EB}$

$= 2\left(\overrightarrow{CA} + \overrightarrow{AE} + \overrightarrow{EB} + \overrightarrow{BC}\right)$

$= 2\left(\overrightarrow{CE} + \overrightarrow{EB} + \overrightarrow{BC}\right)$

$= \vec{O}$

$\therefore$ the forces are in equilibrium.

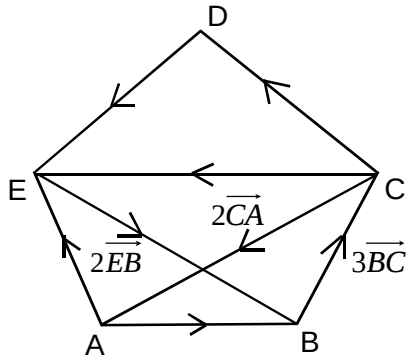


Fig 2.9

CHECK YOUR PROGRESS 2

Ans to Q No 1: P = 6 newton

AD = 9m

DB = 8m

$\therefore$ Now, P. DA = Q. DB

or, 6 x 9 = Q x 8

$\therefore Q = \frac{3\times 9}{4} = \frac{27}{4}$ newton

$\therefore R = \left(6 + \frac{27}{4}\right)$ newton

$= \frac{51}{4}$ newton

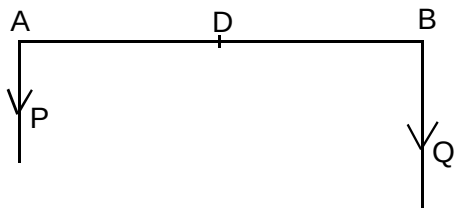


Fig 2.10

Ans to Q No 2: Let P, Q be two like parallel forces acting at A, B respectively.

Let the resultant act at D on AB

$\therefore$ P. AD = Q. BD (1)

Let the positions of the forces be interchanged.

Now, it is given that the resultant acts at D.

$\therefore$ Q. AD = P. BD (2)

From (1) and (2),

$\frac{P}{Q} = \frac{Q}{P}$

i.e., $P^2 = Q^2$

i.e., P = Q

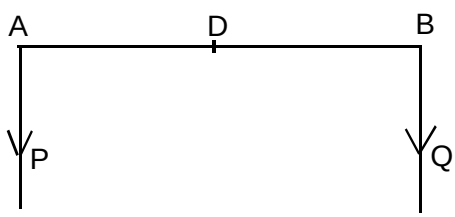


Fig 2.11

Ans to Q No 3: P = 5 newton

AD = 12 m

AB = 9m

P. DA = Q. DB

or, 5 x 12 = Q x 3

$\therefore Q = \frac{5 \times 12}{3} = 20$ newton

$\therefore$ R = Q – P

= (20 – 5) newton = 15 newton.

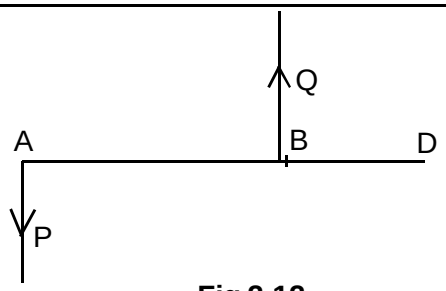


Fig 2.12

Ans to Q No 4: Let P, Q be two unlike parallel forces acting at A, B respectively, where P > Q.

Let the resultant act at D on BA produced.

$\therefore$ P. AD = Q. BD

or P. AD = Q.(AB + AD)

$\therefore$ (P – Q) AD = Q. AB

$\therefore AD = \frac{Q}{P-Q} AB$ (1)

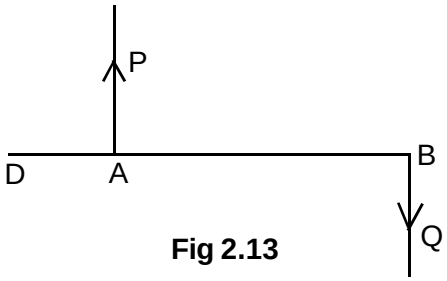


Fig 2.13

Let the magnitude of each force be increased by x.

Let the new resultant act at E on BA produced

$\therefore$(P + x). AE = (Q + x). BE

or, (P + x) AE = (Q + x) (AB + AE)

or, (P + x – Q – x) AE = (Q + x) . AB

$\therefore$ (P – Q). AE = (Q + x). AB

$\therefore AE = \frac{Q+x}{P-Q}.AB$

Now, $\frac{Q+x}{P-Q}.AB = \frac{Q}{P-Q}.AB + \frac{x}{P-Q}.AB > \frac{Q}{P-Q} AB$

i.e. AE > AD.

Ans to Q No 5: Left as exercise

6 (iii) Let I be the incentre of the triangle ABC, where AD is the bisector of the angle BAC.

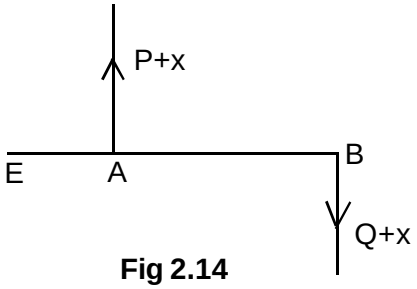


Fig 2.14

$\therefore \frac{AB}{AC} = \frac{BD}{DC}$ (1)

It is given that the resultant passes through the incentre I.

$\therefore$ the resultant of Q at B and R at C must pass through D.

$\therefore$ Q. BD = R. DC, i.e., $\frac{Q}{R} = \frac{DC}{BD}$ (2)

$\therefore$ from (1) and (2)

$\frac{Q}{R} = \frac{AC}{AB}$ i.e. $\frac{Q}{CA} = \frac{R}{AB}$

Similarly, $\frac{P}{BC} = \frac{Q}{CA}$

$\therefore \frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}$

Ans to Q No 6: (i), (ii), (iv), Left as exercise.



2.12 MODEL QUESTIONS

Q 1: Three coplanar forces can act at a point in any manner. Can they produce equilibrium if their magnitudes are (i) 2, 3, 5 (ii) 3, 4, 5 (iii) 1, 3, 5 ?

Q 2: Forces P, Q, R act at the orthocentre O of Δ ABC along OA, OB, OC respectively and are in equibrium. Show that P : Q : R = a : b : c.

Q 3: Forces P, Q, R acting on a particle are in equibrium. If the angle between P and Q is double the angle between P and R. Show that $R^2 = Q(Q - P)$.

Q 4: Two like parallel forces P and Q act at A and B respectively and their resultant R acts at C on AB. Find :

(i) R and AC if P = 12 newton, Q = 8 newton, AB = 10m

(ii) Q and AB if P = 13 newton, R = 20 newton, AC = 14m

(iii) P and Q if R = 12 newton, AC = 0.5m, AB = 3m.

Q 5: Two unlike parallel forces P and Q act at A and B respectively and

their resultant R acts at C on AB (produced). Find :

(i) Q and R if P = 5 newton, AB = 9m, AC = 12m

(ii) P and Q if AB = 2.5m, R = 5 newton, AC = 5m

(iii) AB and R if P = 910 newton, Q = 210 newton, AC = 3m.

Q 6: Two like parallel forces P and Q (P > Q) act at A and B respectively. If their positions are interchanged, prove that the resultant is displaced on AB through a distance $\frac{P-Q}{P+Q}$ AB.

Q 7: Two unlike parallel forces P and Q (P > Q) act at two points x meter apart. If the direction of the greater force is reversed, prove that the position of the resultent is displaced through a distance $\frac{2PQx}{P^2-Q^2}m$.

*** ***** ***

UNIT 3: MOMENT AND COUPLES

UNIT STRUCTURE

3.1 Learning Objectives

3.2 Introduction

3.3 Moment of a force about a point

3.4 Couples

3.5 Let Us Sum Up

3.6 Further Readings

3.7 Answers To Check Your Progress

3.8 Model Questions

3.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- moment of a force about a point
- couples

3.2 INTRODUCTION

In this unit, we will study moments and couples.

3.3 MOMENTS OF A FORCE ABOUT A POINT

The moment of a force about a given point is defined to be the product of the force and the perpendicular distance from the point to the line of action of the force.

Let P be a force acting on a rigid body. Let it tend to rotate the body about the fixed point O. Then the moment of P about the point O = pP.

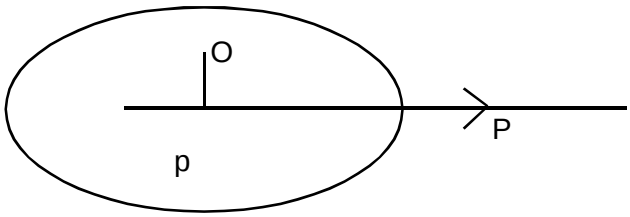


Fig 3.1

Moment is a vector quantity.

The moment of a force about a point is considered to be positive if the tendency of rotation is in the counter-clockwise direction and negative if the tendency of rotation is in the clockwise direction.

Units of Moment : The units of moment are expressed as dyne-cm, newton-m, poundal-ft.

Geometrical Meaning of moment of a force about a point

Let P be a force acting on a rigid body.
Let it tend to rotate the body about the fixed
point O.

Let AB represent the force P.

Moment of P about O

= pP.

Again, area of the triangle OAB

$$= \frac{1}{2} pP$$

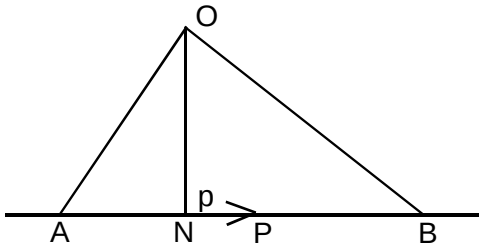


Fig 3.2

$\therefore$ Moment of P about O

= 2 (area of the triangle OAB).

Thus, the moment of a force about a point is twice the area of the triangle whose base is the line-segment representing the force and whose vertex is the point about which the moment is taken.

Example 3 : ABCD is a square of side 4m. Force 5, 6, 7, 8 newton act along the sides AB, CB, CD, AD. Find the sum of their moments about A, B, C, D.

Solution : Sum of the moments about A

= (0 x 5 – 4 x 6 + 4 x 7 x 0 x 8) newton - m

= 4 newton - m

Sum of the moments about B

= (0 x 5 + 0 x 6 + 4 x 7 – 4 x 8) newton - m

= – 4 newton - m

Sum of the moments about C

= (0 x 6 + 0 x 7 – 4 x 8 + 4 x 5)

= – 12 newton - m

Sum of the moments about D

= (0 x 7 + o x 8 + 4 x 6) newton - m

= – 4 newton - m

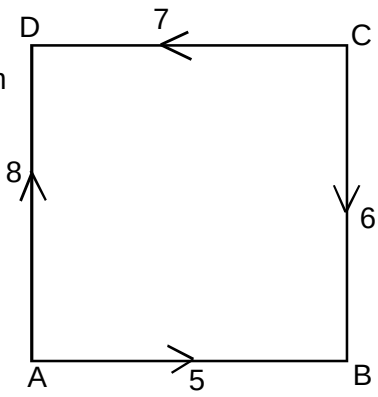


Fig 3.3

Example 4 : ABC is an isosceles triangle board, right angled at C. It is placed on a smooth table. A force 2 newton acts at C along CA and a force 3 newton acts at B along CB. If AC = BC = 1m, find the sum of the moments about A, B and D, where D is the middle point of AB.

Solution : D is the middle point of AB

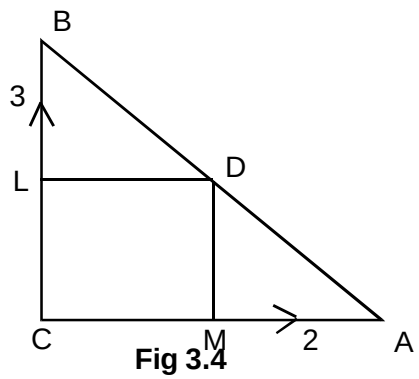


Fig 3.4

$$\therefore DL = \frac{1}{2}CA = \frac{1}{2}m$$

$$DM = \frac{1}{2}BC = \frac{1}{2}m$$

Sum of the moments about A

= (0 x 2 – 1 x 3) newton – m = –3 newton -m

Sum of the moments about B

= (1 x 2 + 0 x 3) newton – m = 2 newton -m

Sum of the moments about D

$= \left(\frac{1}{2}\times 2 - \frac{1}{2}\times 3\right)$ newton m = –0.5 newton -m



CHECK YOUR PROGRESS 3

Q 1: ABCD is a square of side 4m. Forces 4, 3, 6, 10 newton act along AB, BC, DC, DA respectively. Find the algebraic sum of the moments of the forces about A, B, C, D, O, where O is the centre of the square.

Q 2: ABCD is a square of side 4m. Forces 4, 3, 2, 10 newton act along CB, BA, DA, DB respectively. Find the algebraic sum of the moments of the forces about C.

Q 3: ABCD is a square of side 4 m. Force 3, 5, 3, 5 newton act along AB, CB, DC, DA respectively. Find the algebraic sum of the moments of the forces about B, C and O, where O is the centre of the square.

3.4 COUPLES

Two equal and unlike parallel forces (having different lines of action) acting on a rigid body form a couple.

The perpendicular distance between the forces of a couple is called the arm of the couple.

The product of one of the forces of a couple and its arm is called the moment of the couple.

The unit of a couple is the same as that of a moment.

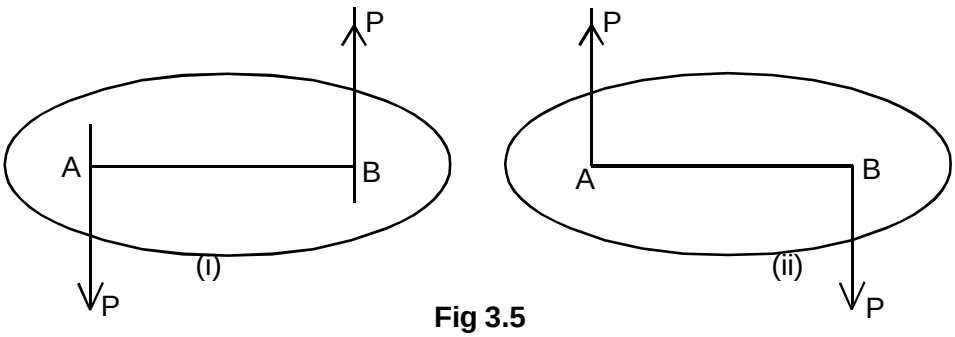


Fig 3.5

Let a couple consist of two forces of magnitude P. Let AB(=p) be the perpendicular distance between the forces. The couple is generally written as (P, P).

The moment of a couple is considered to be positive if the tendency of rotation is in the counter-clockwise direction and negative if the tendency of rotation is in the clockwise direction.

Now, we state a theorem, without proof, which is required to obtain the resultant of any number of couples in a plane.

Theorem : The resultant of two couples acting on a right body in the same plane is couple whose moment is the algebraic sum of the moments of the couples.

Example 5 : Forces equal to 3, 5, 3, 5 newton respectively act along the sides of a square taken in order. Find their resultant.

Solution : Let a be the length of each side of the square.

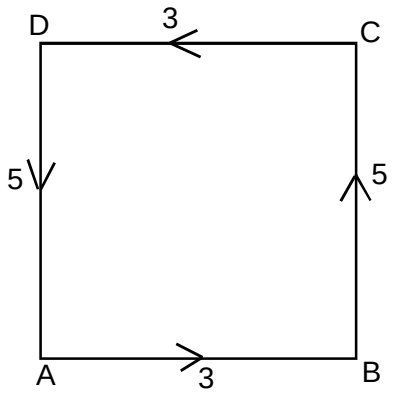


Fig 3.6

From the figure, we see that the force 3 newton along AB and 3 newton along CD form a couple. Its moment is 3a.

Similarly, the force 5 newton along BC and 5 newton along DA form a couple whose moment is 5a.

$\therefore$ the resultant of the given system of forces is (3a + 5a), i.e., 8a.

Example 6 : Forces equal to 8, 7, 8, 7 newton respectively act along the sides of a square taken in order. Find their resultant.

(Left as an exercise)



3.5 LET US SUM UP

- The moment of a force about a given point is defined to be the product of the force and the perpendicular distance from the point to the line of action of the force.
- The moment of a force is considered to be positive if the tendency of rotation is in the counter-clockwise direction and negative if the tendency of rotation is in the clockwise direction.
- Two equal and unlike parallel forces (having different lines of action) acting on a rigid body form a couple.

  The perpendicular distance between the forces of a couple is called the arm of the couple.

  The product of one of the forces of a couple and its arm is called the moment of the couple.



3.6 FURTHER READING

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridge University Press.

2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and Ashok Gupta.

3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and Sankara subramanian. Vikas Publishing House Pvt. Ltd.



3.7 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS

Ans to Q No 1: **Q** O is the centre of the square ABCD, therefore its distance from the sides of the square $= \frac{1}{2} \times 4m = 2m$

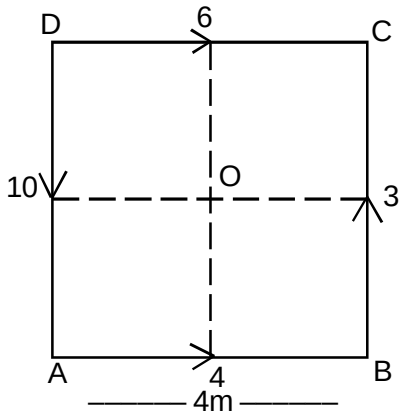


Fig 3.7

Sum of the moments about A

= 0 x 4 + 4 x 3 – 4 X 6 + 0 x 10 = –12 newton m

Sum of the moments about B

= 0 x 4 + 0 x 3 – 4 x 6 + 4 x 10 = 16 newton m.

Sum of the moments about C

= 0 x 3 + 0 x 6 + 4 x 10 + 4 x 4

= 56 newton m

Sum of the moments about D = 0 x 6 + 0 x 10 + 4 x 4 + 4 x 3

= 28 newton m

Sum of the moments about O = 2 x 4 + 2 x 3 – 2 x 6 + 2 x 10

= 22 newton m

Ans to Q No 2: Left as exercise.

Ans to Q No 3: Left as exercise.



3.8 MODEL QUESTIONS

Q 1: ABCD is a square of sides 1.5 m each. Forces of magnitude 4, 3, 2 and $5\sqrt{2}$ act along CB, BA, DA and DC respectively. Find the sum of the moments of the forces about C.

Q 2: Forces having magnitudes 4, 7, 4 and 7 units act along the sides of a

square taken in order, prove that the resultant of the forces is a couple whose moment is of magnitude 11a units, a being the length of each side of the square.

*** ***** ***

UNIT 4 : FRICTION

UNIT STRUCTURE

4.1 Learning Objectives

4.2 Introduction

4.3 Friction

- 4.3.1 Definition
- 4.3.2 Types of Friction
- 4.3.3 Limiting Friction
- 4.3.4 Co-efficient of Friction
- 4.3.5 Laws of Friction

4.4 Let Us Sum Up

4.5 Further Reading

4.6 Answers To Check Your Progress

4.7 Model Questions

4.1 LEARNING OBJECTIVES

After going through this unit, you will able to know

- definition of friction
- types of friction
- limiting friction & co-efficient of friction
- laws of friction.

4.2 INTRODUCTION

In previous units, we have already discussed forces ,types of forces and coplanar forces. in this unit we will study friction, type of friction, limiting friction and co-efficient of friction. In this unit, we will also discuss the laws of friction.

4.2 FRICTION

4.3.1 Definition

If a body slides or rolls on another, there is a force which tends to resist motion. This resistance is called the force of friction. Friction is a force which resist the sliding of one body over another. It is a self adjusting force.

4.3.2 Types of Friction

Friction of the following two types–

1) Statical friction
2) Dynamical friction

1) Statical friction : The force of friction acting on a body at rest is called Statical friction.

2) Dynamical friction : The force of friction a acting on a body when it is actually in motion is called dynamical friction. It is also called kinetic friction. Dynamical friction is of the following two types.

(a) Sliding friction : When a body slides over another body, the friction experienced by the body is called sliding friction.

(b) Rolling friction : When a body rolls over another body the friction experiences by the body is called rolling friction.

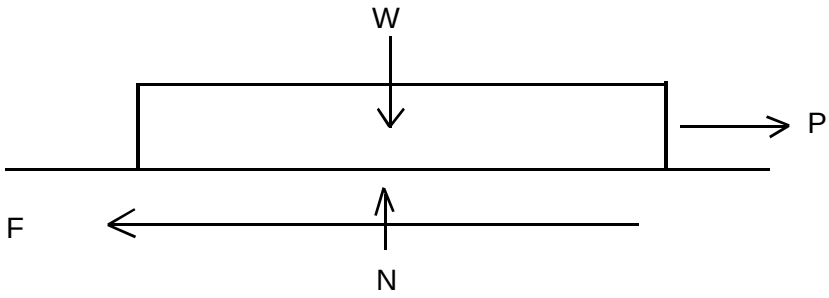
4.3.3 Limiting Friction

The force of friction acting on a body when it is on the point of motion is called limiting friction. limiting friction is equal in magnitude and opposite in direction to the force which tends to produce motion

4.3.4 Co-efficient of Friction

Limiting friction between two bodies bears a constant ratio to the normal reaction between them. This ratio is called the coefficient of friction.

It depend on the nature of the surfaces in contact.



Co-efficience of friction = $\frac{F}{N}$

Where F is limiting friction and N is normal reaction between the contact surfaces.

Co-efficient of friction is denoted by m .

There $m = \frac{F}{N}$

4.3.5 Laws of Friction

1. The direction of the force of friction is always opposite to that in which the point of contact has the tendency to move.
2. The magnitudes of the force of friction is such that it is sufficient to prevent the motion of the point of contact, subject to a certain minimum limit.
3. Limiting friction is independent of the areas in contact.



CHECK YOUR PROGRESS

Q 1: Define friction?

.....

.....

Q 2: Define friction?

.....

.....

Q 3: Define co-efficient of friction?

.....

.....



4.4 LET US SUM UP

- If a body slides or rolls on another, there is a force which tends to resist motion. This resistance is called the force of friction.
- Friction is a force which resists the sliding of one body over another. It is a self adjusting force.
- Friction is of two types : (a) Statical friction (b) Dynamical friction
- Dynamical friction is of two types– (a) Sliding friction (b) Rolling friction.
- The force of friction acting on a body when it is on the point of motion is called limiting friction.
- Limiting friction between the body bears a constant ratio to the normal reaction between them. This ratio is called the co-efficient of friction (m).

  i.e., $m = \frac{F}{N}$

  where F = limiting friction

  N = Normal reaction between the contact surfaces.



4.5 FURTHER READING

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridge University Press.

2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and

Ashok Gupta.

3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and Sankara subramanian. Vikas Publishing House Pvt. Ltd.



4.6 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: See section 4.3.1

Ans to Q No 2: See section 4.3.5

Ans to Q No 3: See Section 4.3.4



4.7 MODEL QUESTIONS

Q 1: Explain statical friction and dynamical friction.

Q 2: Define limiting friction.

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UNIT 5: KINEMATICS OF A PARTICLE

UNIT STRUCTURE

5.1 Learning Objectives

5.2 Introduction

5.3 Kinematics of a Particle and related terms

5.4 Relative velocity

5.5 Angular velocity

5.6 Rectilinear motion under constant acceleration

5.7 Newton's Laws of Motion

5.8 Let Us Sum Up

5.9 Further Readings

5.10 Answers To Check Your Progress

5.11 Model Questions

5.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- kinematics of a particle and related terms
- rectilinear motion under constant acceleration
- Newton's laws of motion.

5.2 INTRODUCTION

Dynamics is a branch of Mechanics. It deals with motion of bodies under the action of forces.

Dynamics is generally divided into two parts, – Kinetics and Kinematics. In this unit, we will study kinematics of a particle and related terms; angular velocity, relative velocity etc. Also, we will study rectilinear motion under constant acceleration. We will also study Newton's laws of motion.

5.3 KINEMATICS OF A PARTICLE AND RELATED TERMS

Kinematics deals with motion of material bodies without taking into consideration of their mass and the forces which cause it.

Matter : Matter is anything that occupies space and is perceptible by our senses.

Mody : A body (or material body) is a limited portion of matter.

Volume : The amount of space occupied by body is called its volume.

Mass : Mass is the quantity of matter in a body.

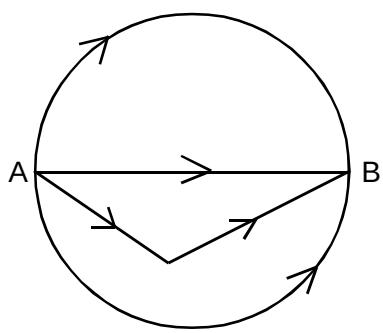
Particle : Particle is a body of negligible mass. It may be identified with a moving geometrical point.

Position : A point at which a particle may be found at a certain time is known as position.

Motion and Rest : Motion is change of position of a body with time. When it is not changing its position, it is said to be at rest.

Displacement : If a particle moves along a certain path, its position changes. The change of position between two points, whatever may be the path, is called displacement.

In the adjoining figure, A is the initial position and B is the final position of the particle. $\overrightarrow{AB}$ is the displacement.



We note that displacement has three characteristics, viz, (a) magnitude – represented by the length AB, (b) directions – along AB, (c) sense, – the displacement from A to B.

Thus, displacement is a vector quantity.

Velocity : The velocity of a moving particle is its rate of change of displacement having a finite direction and magnitude.

Velocity is a vector quantity.

Speed : Rate of change of position is speed. It has no specified direction.

Speed is a scalar quantity.

Note : In rectilinear motion, speed and velocity are identical.

Units of magnitude of velocity : If the displacement of a particle is measured in metre and time in second, the unit of speed is one metre per second. It is denoted by m/s. If the displacement of a particle is measured in kilometre and time in hour ,the unit of speed is one kilometre per hour. It is denoted by km/h.

It may be noted that 18km/h = 5 m/s.

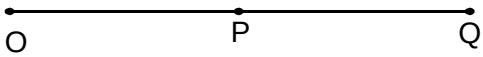
Uniform velocity

If a particle moves along the same straight line in the same sense, and passes over equal distances in equal intervals of time, its velocity is said to be uniform.

Analytical Expression for Velocity

Let a moving point P be at a distance x from a fixed point O at time t. Let the particle be at Q at time $t+dt$, where $OQ = x+dx$

$\therefore PQ = dx$



Velocity u of P at time t = $\lim_{dt \to o} \frac{dx}{dt}$ (**Q**velocity is the rate of change of displacement)

$= \frac{dx}{dt}$

i.e., $u = \frac{dx}{dt}$.

For uniform velocity, $u = \frac{dx}{dt}$ = constant.

For non-uniform velocity, $u = \frac{dx}{dt} \neq$ a constant.

Acceleration

The rate of change of velocity of a moving particle is called its acceleration.

If a particle moves in a straight line and its velocity increases, the acceleration is positive; and if the velocity decreases, the acceleration is negative. A negative acceleration is said to be retardation.

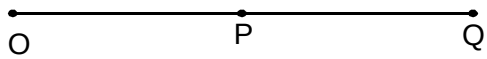
Units of acceleration

Unit of acceleration is one metre or one contimetre per second change of velocity per second. Symbolically it is written as 1m/s^2 or 1 cm/ s^2. The first "per second" gives the rate of change of velocity, and the second "per second" gives the unit of time in which the change of velocity occurs.

Analytical Expression for Acceleration

Let u be the velocity of a particle at time t, and $u+du$ be its velocity at time $t+dt$.

Acceleration f at time t



$= \lim_{dt \to 0} \frac{(u+du)-u}{dt}$ (**Q** acceleration is the rate of change of velocity)

$= \lim_{dt \to 0} \frac{du}{dt}$

$= \frac{du}{dt}$

i.e., $f = \frac{du}{dt}$

Now $f = \frac{du}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

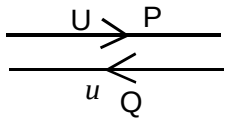
Again $f = \frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} = u\frac{du}{dx}$

Thus, $f = \frac{du}{dt} = \frac{d^2x}{dt^2} = u\frac{du}{dx}$

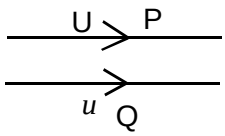
5.4 RELATIVE VELOCITY

Let two points P and Q move in the same place. The rate of change of position of Q as seen from P is defined as the relative velocity of Q with

respect to P.



If P move with velocity u and Q move with velocity u in the opposite direction of P then, velocity of P relative to Q is $= u - (-u) = u + u$



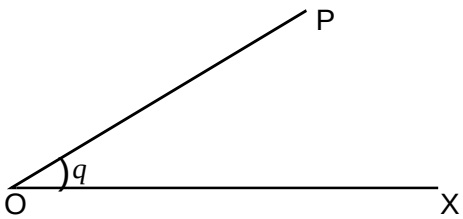
If P and Q move in the same direction with their respective velocities u and u then, velocity of P relative to Q is $= u - u$

5.5 ANGULAR VELOCITY

The rate of change of the angle made by the line joining a fixed point to the moving point on a plane with any straight line drawn in a fixed direction in that plane is called the angular velocity of the moving point about the fixed point.

Let O be a fixed point on a fixed line OX in a plane.

Let P be the position of a particle moving in the plane at time t.



Let $\angle XOP = q$. It is measured positive in the anti-clockwise sense.

The rate of change of q with respect to time t, i.e., $\frac{dq}{dt}$ is the angular velocity of the particle at P about the point O. Thus, the rate at which the angle q is increasing is the angular velocity of the particle about O.

Angular velocity is measured in radians per second. Thus, if w is the angular velocity of a particle, then

$$w = \frac{du}{dt} \text{ rad/s}$$

The rate of change of w with respect to time t, i.e., $\frac{dw}{dt}$, i.e., $\frac{d^2q}{dt^2}$ is the acceleration in the sense of q increasing.



CHECK YOUR PROGRESS - 1

Q 1: A particle, moving in a straight line travels a distance x in time t. If $x = 3t^2 + 2t$, find velocity and acceleration after 4 seconds (x is in metres, t in seconds).

Q 2: A particle, moving in a straight line, travels a distance x in time t (x is in metres, t in seconds). If $x = 9t^2 - t^3 - 4$, find (i) its initial acceleration, (ii) its velocity when its acceleration is zero, (iii) its distance from the starting point when it again comes to rest.

Q 3: A particle moves in a straight line in such a way that its distance from a fixed point O in the line at the end of t seconds is given by $x = t^3 + 18t + 11$ (measured in cm).

Find its distance from O, its velocity and acceleration at the end of 5 seconds.

Q 4: The velocity of a moving particle on a line at a distance x from a fixed point in its line of motion is given by $J^2 = 2(x\sin x + \cos x)$. Find its acceleration.

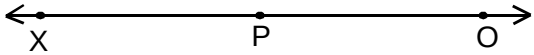
Q 5: The velocity of a moving particle at a distance x from a fixed point in its line of action is given by $v^2 = ax^2 + 2bx + c$, where a, b, c are constants.Prove that the acceleration is proportional to the distance from the fixed point.

5.6 RECTILINEAR MOTION UNDER CONSTANT ACCELERATION

The motion of a particle which takes place along a straight line is known as rectilinear motion.

Let a particle move in a straight line under constant acceleration.

Let OX be the straight line in which the particle moves.



Let the particle be at O at the start of the motion. O is called the initial position of the particle.

Let P be the position of the particle at time t. Let OP = x.

Let the particle start with velocity u and let u be its velocity at P.

Let f be the constant acceleration.

Now, we prove the following relations.

I. $\boxed{v = u + ft}$

We have $f = \frac{du}{dt}$

$\therefore dv - fdt$

$\therefore \int dv = f \int dt$

or $v = ft + c$, where c is a constant of integration.

When t = 0, $v = u$

$\therefore$ u = f0 + c or c = u

$\therefore v = ft + u$ i.e. $\boxed{v = u + ft}$

II. $\boxed{s = ut + \frac{1}{2} f t^2}$

We have $v = u + f t$

Now, $v = \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = u + f t$

or $dx = udt + ftdt$

$\therefore \int dx = u \int dt + f \int t d$

$\therefore x = ut + \frac{ft^2}{2} + c$, where c is a constant

When t = 0, x = 0

$\therefore$ 0 = 0 + 0 + c i.e. c = 0

$\therefore x = ut + \frac{1}{2} f t^2$

or $\boxed{s = ut + \frac{1}{2} f t^2}$

where s is the distance or space (s for space) described by the particle in time t.

III. $\boxed{v^2 = u^2 + 2fs}$

We have

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Also $\frac{dv}{dt} = f$

$\therefore\ v\frac{dv}{dx} = f$

or, $vdv = f\,dx$

$\therefore\ \int vdv = f\int dx$

$\therefore\ \frac{v^2}{2} = fx + c$, where c is a constant of integration.

When $v = u$ x = 0

$\therefore\ \frac{d^2}{2} = 0 + c$ i.e. $C = \frac{u^2}{2}$

$\therefore\ \frac{v^2}{2} = fx + = \frac{u^2}{2}$

or $v^2 = u^2 + 2fx$

or $\boxed{v^2 = u^2 + 2fs}$

IV. Expression for the distance travelled by a particle in n th second of its motion in a straight line.

Let s_n and s_{n-1} denote the distances described by the particle in n seconds and (n–1) seconds respectively.

$\therefore$ the distance travelled in the nth second

= $s_n - s_{n-1}$

$$= \left(un + \frac{1}{2}fn^2\right) - \left(u(n-1) + \frac{1}{2}f(n-1)^2\right)$$

$$= u + \frac{1}{2}f(n-1)$$

Theorem :The average velocity is the mean of the initial and the final velocities and is equal to the velocity at the middle of the interval.

Proof Average velocity $= \frac{total\ distance\ traversed}{total\ time\ taken}$

$$= \frac{s}{t}$$

$$= \frac{ut + \frac{1}{2}ft^2}{t}$$

$$= u + \frac{1}{2}ft$$

$$= \frac{2u + ft}{2}$$

$$= \frac{u + (u + ft)}{2}$$

$$= \frac{u + v}{2}$$

= mean of initial and final velocities.

Again, average velocity $= \frac{s}{t}$

$$= \frac{ut + \frac{1}{2}ft^2}{t}$$

$$= u + \frac{1}{2}ft$$

$$= u + f\left(\frac{t}{2}\right)$$

= Velocity at time $\frac{t}{2}$

= Velocity at the middle of the interval.

Note : If the particle moves under variable acceleration or if the particle moves in a curved path, the above relations do not hold good.

Example 1: A particle moving in a straight line with acceleration 5m/s^2 describes 200 metres in 5 seconds. What are its initial and final velocities?

Solution : Let the initial velocity be u m/s and the final velocity be v m/s.

Now $200 = u5 + \frac{1}{2} \times 5 \times 5^2 \qquad \left(s = ut + \frac{1}{2}ft^2\right)$

or, $200 = 5u + \frac{125}{2}$

or, $10u = 400 - 125$

or, $10u = 275$

$\therefore u = 27.5$

Againg $v = 27.5 + 5 \times 5$ $(v = u + ft)$

= 52.5

$\therefore$ initial velocity = 27.5 m/s

final velocity = 52.5 m/s

Example 2 : A particle is moving in a straight line with uniform acceleration. In the 11th and 15th seconds from the commencement, it moves through 720 and 960 cms respectively. Find its initial velocity and the acceleration with which it moves.

Solution : Let the initial velocity and the acceleration be u cm/s and f cm/ s^2 respectivaly.

$$\therefore\ 720 = u + \frac{1}{2} f(2 \times 11 - 1)$$

or $$720 = u + \frac{21}{2} f \qquad (1)$$

and $$960 = u + \frac{1}{2} f(2 \times 15 - 1)$$

or $$960 = u + \frac{29}{2} f \qquad (2)$$

Subtracting (1) from (2), we get

240 = 4f or f = 60

$\therefore$ from (1), $720 = u + \frac{21}{2} \times 60$

$\therefore$ u = 720 – 630 = 90

$\therefore$ initial velocity = 90 cm/s

acceleration = 60 cm/s^2

Example 3 : Prove that, for a particle moving with uniform acceleration f in a straight line,

$$f = 2\left(\frac{s'}{t'} - \frac{s}{t}\right) / \left(t + t'\right)$$

where s is the space deseribed in t seconds and s/ during the next t/ seconds.

Solution : Let u be the initial velocity and v be the velocity after time t.

$$\therefore\ v = u + f t \qquad (1)$$

$$s = ut + \frac{1}{2} f t^2 \qquad (2)$$

$$s' = vt' + \frac{1}{2} f t'^2$$

i.e. $s' = (u + ft)t' + \frac{1}{2} f t'^2 \qquad (3)$

From (2), $\frac{s}{t} = u + \frac{1}{2} f t \qquad (4)$

From (3), $\frac{s'}{t'} = u + ft + \frac{1}{2} f t' \qquad (5)$

Subtracting (4) from (5), we get

$$\frac{s'}{t'} - \frac{s}{t} = \frac{1}{2} ft + \frac{1}{2} f t'$$

or, $$2\left(\frac{s'}{t'} - \frac{s}{t}\right) = f\left(t + t'\right)$$

$$\therefore \quad f = 2\left(\frac{s'}{t'} - \frac{s}{t}\right) / \left(t + t'\right)$$



CHECK YOUR PROGRESS - 2

Q 1: A particle which is moving along a straight line at the rate of 60 m/s is brought to rest in 3 minutes with uniform retardation. Find the retardation and also the distance that the particle travels before coming to rest. [Retardation is negative acceleration. When velocity decreases, its rate of change is negative.]

Q 2: A bullet fired into a target loses half its velocity after penetrating 3 cm. How much further will it penetrate?

Q 3: If x_1 and x_2 be the spaces described by a particle moving on a straight line during the pth and (p+q)th seconds of its motion respectively, show that its acceleration is $\frac{x_2 - x_1}{q}$.

Q 4: A train moving on a straight path and stopping at two stations 2 km apart takes 4 minutes on the journey from one of the

stations to the other. Assuming that its motion is first that of uniform acceleration a and then that of uniform retardation b, prove that $\frac{1}{a}+\frac{1}{b}=4$

Q 5: A particle starts with initial velocity u to move in a straight line and passes over the first half of the given distance with a uniform acceleration f_1 and over the second half with uniform acceleration f_2. Show that the final velocity is the same as if the whole distance was traversed with uniform acceleration $\frac{1}{2}(f_1 + f_2)$.

Q 6: If a particle moving on a straight line with uniform acceleration describes successive equal distances in times t_1, t_2, t_3 respectively, show that $\frac{1}{t_1}-\frac{1}{t_1}+\frac{1}{t_3}=\frac{3}{t_1+t_1+t_3}$.

5.7 NEWTON'S LAWS OF MOTION

Sir Isaac Newton (1642-1727) formulated the following statements. These are known as Laws of Motion.

First Law : Every body continues in its state of rest or of uniform motion in a straight line unless acted on by some external force.

Second Law : The rate of change of momentum is proportional to the applied force and takes place in the direction in which the force acts.

Third Law : To every action, there is an equal and opposite reaction.

The above laws are taken as axioms. These axioms are based on experience and observations. There are no direct proofs of these laws.

Momentum

The momentum of a moving particle at any instant is the product of its mass and its velocity at that instant.

The momentum of a body is a vector quantity.

Unit of momentum : The unit of momentum is defined as "unit mass moving with unit velocity".

In CGS system, the unit of momentum is gm-cm per sec.

In FPS system, the unit of momentum is lb-ft per sec.

If a body of mass m gm moves with a velocity u cm/s, the momentum is mu gm-cm/s

The formula P = mf

Let a force P act on a body of mass m. Let the force change the velocity of the body continuously.

Let v be the velocity and f the acceleration produced at any instant t.

Then rate of change of momentum

$$= \frac{d}{dt}(mv)$$

$$= m\frac{dv}{dt} + v\frac{dm}{dt}$$

= mf [For constant mass m, $\frac{dm}{dt} = 0$]

The above acceleration is along the line of action of P.

By Newton's second Law of Motion, P is proportional to mf.

$\therefore$ P = kmf, where k is a constant

We choose a unit of force so that k = 1.

Let unit force be that force which acting on unit mass produces unit acceleration. Thus, we have

P = 1, if m = 1 and f = 1

$\therefore$ k = 1

$\therefore$ P = mf

Units of Force

Newton's Second Law of Motion gives us a method of measuring forces by applying them to a standard mass and observing their applications.
1 Newton is that force which gives a mass of 1 kg an acceleration of 1 m/s^2.
1 dyne is that force which gives a mass of 1 gm an acceleration of 1 cm/s^2.
1 poundal is that force which gives a mass of 1 lb an acceleration of 1 ft/s^2.

The above units, viz, Newton, dyne and poundal are known as absolute units of force. They do not depend upon the value of g, the acceleration due to gravrty.

The weight of a body is the force with which the earth attracts the body.

Let W be weight of a mass m. The acceleration produced by the force W in the mass m is the acceleration of due to granty.

Using the formula P = mf, we get W = mg.

1. When the mass is 1 gm, the weight is g dyne.
2. When the mass is 1 kg, the weight is g newton.
3. When the mass is 1 lb, the weight is g poundal.

We have the following relations

1 gm wt = g dyne (g = 981)

1 kg wt = g newton (g = 9.81)

1 lb wt = g poundal (g = 32.2)

The above units, viz, gm wt, kg wt, lb wt are called gravitational units of force. They depend on the value of g, the acceleration due to g.

Note : I newton = 1 kg x 1 m/s^2 = 10^3 gm x 10^2 cm/s^2 = 10^5 dyne

1 poundal = 13800 dyne = 0.138 newton.

Example 1 : What is the weight of a body if its mass is 1 gm ?

Solution : The required weight

= 981 dyne

Example 2 : What is the weight of a body if its mass is 1 kg ?

Solution : The required weight

= 9.81 N

Example 3 : What is the mass of a body if its weight is 29.4 N ?

Solution : The required mass

$\frac{29.4}{9.8}$ kg = 3 kg (g = 9.8)

Example 4 : A force of 8N acts on a body of mass 2 kg. What is the acceleration ?

Solution : Here P = 8N, m = 2kg

Using the formula P = mf, we have

$$f = \frac{p}{m} = \frac{8}{2} m/s^2 = 4m/s^2$$



CHECK YOUR PROGRESS - 3

Q 1: A mass of 5 gm is accelerating at 981 cm/ s^2. What is the force acting on it ?

Q 2: A particle of mass 675 g is accelerating at 16 cm/s^2. Find the force acting on it in newton.

Q 3: A body of moves 45 kg is accelerating at 6m/s^2. Find the force acting on it.

Q 4: Find the weight of mass of 2 ton on the earth. (1 ton = 1000 kg)

Q 5: A particle of mass 20 g is acted upon by a constant force, which produces a velocity of 15 cm/s in 5 seconds. If the particle is initially at rest, find the magnitude of the force.

Q 6: A uniform force of 520 dyne, acting for half a minute, changes the velocity of a body moving in a straight line from 2.9 m/s to 3.5 m/s. Find the mass of the body.

Q 7: A constant force of 200 dyne changes the velocity of a body moving in a straight line from 250 m/s to 300 m/s in one minute. Find the mass of the body.

Q 8: A mass m gram is acted on by a constant force of P dyne under which, in t second, it moves a distance x cm from rest, and acquires a velocity v cm per second. Show that $x = \frac{1}{2}\frac{mv^2}{p}$



5.8 LET US SUM UP

- If a particle moves along a certain path, its position changes. The change of position between two points, whatever may be the path, is called displacement. Displacement is a vector quantity.
- The velocity of a moving particle is its rate of change of displacement having a finite direction and magnitude.

Velocity is a vector quantity.

- Rate of change of position is speed. It has no specified direction. Speed is a scalar quantity.
- In rectilinear motion, speed and velocity are identical.
- If a particle moves along the same straight line in the same sense, and passes over equal distances in equal intervals of time, its velocity is said to be uniform.
- The rate of change of velocity of a moving particle is called its acceleration.
- A negative acceleration is said to be reardation.
- Let two points P and Q move in the same plane. The rate of change of position of Q as seen from P is defined as the relative velocity of Q with respect to P.
- The rate of change of the angle made by the line joining a fixed point to the moving point on a plane with any straight line drawn in a fixed direction in that plane is called the angular velocity of the moving point about the fixed point.

  Angular velocity is measured in radian per second.
- The motion of a paiticle which takes place along a straight line is known as rectilinear motion.
- Let a particle start with a velocity u and move along a straight line with uniform acceleration f. Let s be the displacement at the end of time t, and n be the velocity at that time. Then

  1. $v = u + ft$
  2. $s = ut + \frac{1}{2} ft^2$
  3. $v^2 = u^2 + 2fs$
- The average velocity is the mean of the initial and the final velocities and is equal to the velocity at the middle of the interval.
- Newtons Laws of Motion are as follows :

  First Law : Every body continues in its state of rest or of uniform motion in a straight line unless acted on by some external force.

  Second Law : The rate of change of momentum is proportional to the applied force and takes place in the direction in which the force acts.

Third Law : To every action, there is an opposite and equal reaction.

- The momentum of a moving particle at any instant is the product of its mass and its velocity at that instant.
- P = mf, where m is the mass of a body, P the applied force acting on it and f the acceleration produced by P.
- P = mf is satisfied when

(1) P is in newton, m in kg, f in m/s^2

or (2) P is in dyne, m in gram, f in cm/s^2

or (3) P is in poundal, m in lb, f in ft/s^2.



5.9 FURTHER READING

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridge University Press.

2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and Ashok Gupta.

3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and Sankara subramanian. Vikas Publishing House Pvt. Ltd.



5.10 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS 1

Ans to Q No 1: Here $x = 3t^2 + 2t$

$$\therefore\ v = \frac{dx}{dt} = 6t + 2$$

$$f = \frac{d^2x}{dt^2} = 6$$

∴ velocity after 4 seconds = (6 x 4 + 2) m/s

= 26 m/s

acceleration after 4 seconds = 6 m/s^2

Ans to Q No 2: Here $x = 9t^2 - t^3 - 4$

$$\therefore\ v = \frac{dx}{dt} = 18t - 3t^2$$

$$f = \frac{d^2x}{dt^2} = 18 - 6t$$

(i) Initial acceleration

= Acceleration (at t = 0)

= (18 – 6 x 0) m/s^2

= 18m/s^2

(ii) When acceleration is zero; $\frac{d^2x}{dt^2} = 0$

i.e. 18 – 6t = 0

i.e. t = 3

$\therefore$ Velocity at t = 3

= (18 x 3 – 3 x 3^2) m/s

= 27m/s

(iii) When the particle comes to rest,

$$v = \frac{dx}{dt} = 0$$

This $\Rightarrow$ $18t - 3t^2 = 0$ $\Rightarrow$ $t(6 - t) = 0$ $\Rightarrow$ t = 0 or t = 6.

When t = 0, x = 0

When t = 6, x = 9 x 6^2 – 6^3 – 4 = 104

$\therefore$ the required distance = 104m

Ans to Q No 3: Here $x = t^3 + 18t + 11$

$$\therefore\ v = \frac{dx}{dt} = 3t^2 + 18$$

$$f = \frac{d^2x}{dt^2} = 6t$$

At the end of 5 seconds,

x = (5^3 + 18 x 3 + 11) cm = 190 cm

v =(3 x 5^2 + 18) cm/s = 93 cm/s

f = 6 x 5 cm/s^2 = 30 cm/s^2.

Ans to Q No 4: Here $n^2 = 2(x\sin x + \cos x)$

$\therefore\ 2v\dfrac{dv}{dx} = 2(1\sin x + x\cos x - \sin x)$

$\Rightarrow\ v\dfrac{dv}{dx} = x\cos x$

$\therefore$ acceleration = x cos x

Ans to Q No 5: Here $v^2 = ax^2 + 2bx + c$

$\therefore\ 2v\dfrac{dv}{dx} = 2ax + 2b$

$\Rightarrow v\dfrac{dv}{dx} = ax + b$

$\therefore$ a and b are constants, therefore acceleration is proportional to the distance from the origin.

CHECK YOUR PROGRESS 2

Ans to Q No 1: Let f m/s^2 be the uniform acceleration and x m be the distance travelled by the particle before coming to rest.

Now 0 = 60 + 180f (v =u + ft, t = 30 x 6 = 180s)

or $f = -\dfrac{1}{3}$

Q f is negative, $\therefore$ the retardation is $\dfrac{1}{3}$ m/s^2.

Again $0^2 = 60^2 + 2.\left(-\dfrac{1}{3}\right)x$

or x = 5400

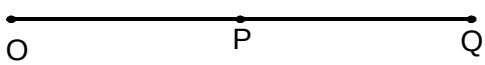
$\therefore$ the required distance is 5400 m

Ans to Q No 2: Let the bullet be fired with velocity u. Let OA = 3 cm

$\therefore$ Velocity at $A = \dfrac{1}{2}u$

Now, using the formula $v^2 = u^2 + 2fs$ we have $\left(\dfrac{u}{2}\right)^2 = u^2 + 2f3$

or, $-\dfrac{3u^2}{4} = 6f$



$\therefore\ f = -\dfrac{u^2}{8}$

Let the bullet penetrate upto B, i.e., will come to rest at B, i.e., velocity at B is zero.

$\therefore$ using the formula $v^2 = u^2 + 2fs$, we have

$0 = \left(\frac{u}{2}\right)^2 + 2\left(-\frac{u^2}{8}\right)s$, where AB = s

or, $\frac{u^2}{4} = \frac{u^2}{4}s$

$\therefore\ s = 1$

$\therefore$ the required distance is 1 cm.

Ans to Q No 3: Let the acceleration be f, and the initial velocity be u.

$\therefore\ x_1 = u + \frac{1}{2}f(2p-1)$ (1)

$x_2 = u + \frac{1}{2}f(2p+2q-1)$ (2)

Subtracting (1) from (2), we get

$x_2 - x_1 = \frac{1}{2}f.2q$

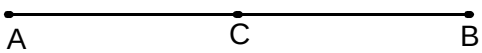
$\therefore\ f = \frac{x_2 - x_1}{q}$

Thus, the required acceleration is $\frac{x_2 - x_1}{q}$.

Ans to Q No 4: The train stops at both the stations A and B.

$\therefore$ the initial velocity as well as the final velocity is zero

We have AB = 2



Let the train travel AC (= s_1) in time t_1 with uniform acceleration a and CB (= s_2) in time t_2 with uniform retardation b.

Clearly $s_1 + s_2 = 2$,

$t_1 + t_2 = 4$

Let the velocity at C be v.

$\therefore\ v = 0 + at_1$ i.e. $t_1 = \frac{v}{a}$

$0 = v - bt_2$ i.e. $t_2 = \frac{v}{b}$

$\therefore\ t_1 + t_2 = v\left(\frac{1}{a} + \frac{1}{b}\right)$

i.e. $v\left(\frac{1}{a} + \frac{1}{b}\right) = 4$ _____(1)

Again, $v^2 = 0 + 2as_1$ i.e. $s_1 = \frac{v^2}{2a}$

$0 = v^2 - 2bs_2$ i.e. $s_2 = \frac{v^2}{2b}$

$\therefore\ s_1 + s_2 = \frac{v^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$

i.e. $\frac{v^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = 2$

i.e. $v^2\left(\frac{1}{a} + \frac{1}{b}\right) = 4$ _____(2)

From (1) and (2), we get

$v = 1$

$\therefore$ from (1), $\frac{1}{a} + \frac{1}{b} = 4$

Ans to Q No 5: Let the given distance be s. Let v be the final velocity and v_1 be the velocity at the end of the first half.

$\therefore\ v_1^{\,2} = u^2 + 2f_1.\frac{s}{2} = u^2 + f_1 s$

$v^2 = v_1^{\,2} + 2f_2.\frac{s}{2}$

$= u^2 + f_1 s + f_2 s$

$= u^2 + (f_1 + f_2)s$

$= u^2 + 2\times\frac{1}{2}(f_1 + f_2)s$

Thus, the result is proved.

Ans to Q No 6: Let u be the velocity at the beginning of t_1, and let f be the acceleration.

$\therefore$ at the beginning of t_2, the velocity is $u + ft_1$.

Again, at the beginning of t_3, the velocity is $(u + ft_1) + ft_2$ i.e. $u + f(t_1 + t_2)$

Let the equal distance be s.

$\therefore\ s = ut_1 + \frac{1}{2} ft_1^{\,2}$ (1)

$s = (u + ft_1)t_2 + \frac{1}{2} ft_2^{\,2}$ (2)

$s = u + f(t_1 + t_2)t_3 + \frac{1}{2} ft_3^{\,2}$ (3)

Also $s + s + s = u(t_1 + t_2 + t_3) + \frac{1}{2} f(t_1 + t_2 + t_3)^2$

i.e. $3s = u(t_1 + t_2 + t_3) + \frac{1}{2} f(t_1 + t_2 + t_3)^2$ (4)

From (1), $\frac{s}{t_1} = u + \frac{1}{2} ft_1$

From (2), $\frac{s}{t_2} = u + ft_1 + \frac{1}{2} ft_2$

From (3), $\frac{s}{t_3} = u + f(t_1 + t_2) + \frac{1}{2} ft_3$

From (4), $\frac{3s}{t_1 + t_2 + t_3} = u + \frac{1}{2} f(t_1 + t_2 + t_3)$

Clearly $\frac{s}{t_1} - \frac{s}{t_2} + \frac{s}{t_3} = \frac{3s}{t_1 + t_2 + t_3}$

i.e. $\frac{1}{t_1} - \frac{1}{t_2} + \frac{1}{t_3} = \frac{3}{t_1 + t_2 + t_3}$

CHECK YOUR PROGRESS 3

Ans to Q No 1: Here m = 5 gm,

f = 981 cm/s^2

$\therefore$ P = 5 x 981 dyne

= 4905 dyne

Ans to Q No 2: Here m = 675 g

f = 16 cm/s^2

$\therefore$ P = 675 x 16 dyne

= 10800 dyne

We know that

10^5 dyne = 1 newton

$\therefore$ 10800 dyne $= \frac{10800}{10^5}$ newton

$= 0.108$ newton

$\therefore$ P = 0.108 newton

Ans to Q No 3: Here m = 45 kg

f = 6 m/s^2

$\therefore$ P = 45 x 6 newton

= 270 newton

Ans to Q No 5: Let f cm/s^2 be the acceleration.

Now from the result $v = u + ft$ we have

16 = 0 + f5

$\therefore$ f = 3 cm/s^2

Also m = 20 g

$\therefore$ P = 20 x 3 dyne

= 60 dyne

Ans to Q No 6: Change in velocity while moving for $\frac{1}{2}$ minute = (3.5 – 2.9)

m/s = 0.6 m/s

$\therefore f = \frac{2 \times 0.6 \times 100}{60} cm/s^2$

= 2 cm/s^2

P = 520 dyne (given)

$\therefore m = \frac{P}{f} = \frac{520}{2} g = 260g$

Ans to Q No 8: We have $v^2 = 0 + 2fx$, where f is the acceleration

(using $v^2 = u^2 + 2fs$)

$\therefore f = \frac{v^2}{2x}$

Againg P = mf

or, $P = m\frac{v^2}{2x}$

$\therefore x = \frac{mv^2}{2p}$



5.11 MODEL QUESTIONS

Q 1: A particle moves along a otraight line such that its displacement x from a fixed point on the line at time t is given by $x = t^3 - 9t^2 + 24t + b$. Determine (i) the instant when the acceleration becomes zero, (ii) The velocity of the particle at that instant.

Q 2: If the distance s is given by $s = at^2 + bt + c$, where t is the time and a, b, c are constants, prove that $4a(s-c) = v^2 - b^2$, where v is the velocity.

Q 3: A point moves with uniform acceleration and v_1, v_2, v_3 donote the average velocities in three successive. intervals t_1, t_2, t_3; prove that $\frac{v_1 - v_2}{v_2 - v_3} = \frac{t_1 - t_2}{t_2 + t_3}$.

Q 4: A particle moving in a straight line is at a distance of x meter from a given point on the line after t seconds from start where $x = t^3 - 2t - 16$. Find its velocity at the end of 3 seconds and acceleration of the end of 4 seconds.

Q 5: A particle moves in a straight line according to the law $n = x^2$. Show that, its acceleration is $2v^{3/2}$.

Q 6: A particle starts from rest and moves in a straight line first with uniform acceleration 'a' and then with uniform reterdation 'b'. If it comes to rest in time t after describing a distance s, prove that $t^2 = 2s\left(\frac{1}{a} + \frac{1}{b}\right)$.

Q 7: What is the mass of a body whose weight is 19.6 N ?

Q 8: What is the magnitude of the force that acts on a mass of 500 gm and produces acceleration $10 m/s^2$?

Q 9: A body of mass m gm at rest is acted upon by a uniform force P gmwt for t seconds. Prove that the velocity aquired is $\frac{pgt}{m} cm/s$ and the

distance described is $\frac{pgt^2}{2m}cm$.

Q 10: A force of 8000 dyne acting on a body increases its velocity from 20 cm/s to 50m/s in 6 secs. Find the mass of the body.

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UNIT 6: SIMPLE HARMONIC MOTION AND RELATED PROBLEMS

UNIT STRUCTURE

6.1 Learning Objectives

6.2 Introduction

6.3 Simple Harmonic Motion

6.3.1 Geometrical Representation of Simple Harmonic Motion

6.3.2 Motion of a Simple Pendulum

6.4 Let Us Sum Up

6.5 Further Readings

6.6 Answers To Check Your Progress

6.7 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will able to learn

- the equation of simple harmonic motion
- the velocity of the particle under simple harmonic motion
- position of the particle moving under simple harmonic motion, periodic time, frequency, amplitude etc.
- geometrical representation of simple harmonic motion
- the motion of a simple pendulum.

6.2 INTRODUCTION

Sometimes we find special types of motion, viz, the motion of a simple pendulum oscillating through a very small angle, the motion of a body tied to an extremity of a spring or elastic spring the other extremity of which being fixed etc. Such motions are simple harmonic motion. Here acceleration is considered to be proportional to the distance.

6.3 SIMPLE HARMONIC MOTION

If a particle moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance from the point, then the particle is said to execute Simple Harmonic Motion (S.H.M).

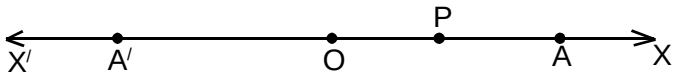


Fig 6.1

Let O be the fixed point on the line X/OX. Let a particle start from rest at A. Let P be the position of the particle at any time t, where OP = x since P is taken on the positive side, therefore x > o.

Then $\frac{dx}{dt}$ is the velocity and $\frac{d^2x}{dt^2}$ is the acceleration of the particle at time t acting in the direction in which x increases.

Since the acceleration is directed towards the fixed point O and is proportional to the distance of the particle from O, hence the equation of motion is $\frac{d^2x}{dt^2} = -mx$ – (1)

where m is a positive constant.

The equation (1) is true whether P lies on the right or left of O.

Let v be the velocity of the particle at time t. Therefore,

$$v = \frac{dx}{dt}, \ \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$\therefore v\,\frac{dv}{dx} = \frac{d^2x}{dt^2}$$

$\therefore$(1) can be written as

$$v\,\frac{dv}{dx} = -m\,x$$

or, $v\,dv = -m\,x dx$

$$\therefore \int v dv = -\int m\,x dx$$

or $\frac{v^2}{2} = -\frac{m\,x^2}{2} + c$ where C is a constant of integration.

Let OA = a

Since the particle starts from rest at A, therefore $v = 0$ when x = a

$$\therefore 0 = -\frac{m a^2}{2} + C$$

i.e. $$c = \frac{m a^2}{2}$$

$$\frac{v^2}{2} = -\frac{m x^2}{2} + \frac{m a^2}{2}$$

i.e. $v^2 = m(a^2 - x^2)$

$$\therefore v = -\sqrt{m(a^2 - x^2)} \quad (2)$$

We note that x decreases when t increases. Therefore $v = \frac{dx}{dt}$ is taken to be negative.

From (2), we have

$$\frac{dx}{dt} = -\sqrt{m}\ \sqrt{a^2 - x^2}$$

or $$\sqrt{m}\ dt = -\frac{1}{\sqrt{a^2 - x^2}} dx$$

$$\therefore \int \sqrt{m}\ dt = \int \left(-\frac{1}{\sqrt{a^2 - x^2}} \right) dx$$

or $\sqrt{m}\ dt = \cos^{-1}\frac{x}{a} + k$, where k is a constant of integration.

Since the particle starts from rest at A, therefore t = 0 when $x = a$

$$\therefore 0 = \cos^{-1}\frac{a}{a} + k$$

i.e. $k = 0$

$$\therefore \sqrt{m}\ t = \cos^{-1}\frac{x}{a}$$

or $x = a \cos\sqrt{m}\ t$ ... (3)

Note :

1. From (2), we get the velocity of the moving particle at any distance x. It is $-\sqrt{m}\ \sqrt{a^2 - x^2}$.
2. From (3), we get the distance of the moving particle from the fixed

point at any time t. It is $a\cos\sqrt{m}\,t$.

3. When x = 0, from (2), we get $n = -\sqrt{m}\,a$. i.e. the velocity of the moving particle at the fixed point O is $-\sqrt{m}\,a$ which is maximum. This implies that the particle does not stop at O. It passes through O on the other side OX/. Let A/ be a point on the line on the other side such that OA/ = OA = a.

Clearly, from (2), $v = 0$ when x = a or x = -a, i.e. the velocity is zero at A or at A/. From A/, the particle retraces its path, passes through O and again it instantaneously comes to rest at A. Thus, the motion of the particle is oscillatory about the point O (from A to A/, and then back from A/ to A). It is repeated over and over again. Such type of motion is known as Simple Harmonic Motion. The point O is known as the centre of oscillation or the centre of mean position.

Amplitude

The maximum distance OA or OA/(=a) to which the particle moves on either side of O is called the amplitude. Thus, the maximum displacement traversed by the particle on either side of the centre of oscillation is called the amplitude.

Periodic Time

The time for one complete oscillation, i.e. from A to A/, and back to A, is called the periodic time or the time of a complete oscillation.

Putting x = 0 in (3), we get

$0 = a\cos\sqrt{m}\,t$ [We note that at the point O, x = 0]

$\therefore \cos\sqrt{m}\,t = 0$

or $\sqrt{m}\,t = \frac{p}{2}$

$\therefore t = \frac{p}{2\sqrt{m}}$

$\therefore$ the time taken by the particle from A to O is $\frac{p}{2\sqrt{m}}$

Now, time from A to O

= time from O to A/

= time from A/ to O

= time from O to A.

$\therefore$ periodic time $= 4 \times \frac{p}{2\sqrt{m}} = \frac{2p}{\sqrt{m}}$

We note that the periodic time $\frac{2p}{\sqrt{m}}$ does not contain a. Thus, the periodic time is independent of the amplitude a.

Frequency

The number of complete oscillations per unit time is called the frequency of oscillation.

Let n denote the frequency any T denote the periodic time.

$\therefore nT = 1$

or $n = \frac{1}{T} = \frac{\sqrt{m}}{2p}$

or $m = 4n^2p^2$

Example 1 : The position of a particle moving in a straight time is given by x = a cos nt + b sin nt. Prove that the particle executes simple harmonic motion. Find the periodic time and the amplitude.

Solution : We have

x = a cos nt + b sin nt.

Differentiating with respect to t, we get

$$\frac{dx}{dt} = -an \sin nt + bn \cos nt$$

Again, differentiating we get

$$\frac{d^2x}{dt^2} = -an^2 \cos nt - bn^2 \sin nt$$

i.e. $\frac{d^2x}{dt^2} = -n^2x$ (1)

From equation (1), we see that the acceleration is proportional to the distance and it is directed towards the fixed point. Therefore, the motion executed by the particle is simple harmonic.

Periodic time $= \frac{2p}{\sqrt{m}} = \frac{2p}{m}$ (Here $m = n^2$)

We have

$$v = \frac{dx}{dt} = -an \sin t + bn \cos nt$$

Again, we know that

amplitude = the distance between the centre of oscillation and the position of instantaueous rest where velocity is zero.

Now, when $v = 0$

–a n sin nt + bn cos nt = 0

or an sin nt = bn cos nt

or $\frac{\sin nt}{bn} = \frac{\cos nt}{an} = k$, (say)

$\therefore$ sin nt = kbn, con nt = kan

Now $\cos^2 nt + \sin^2 nt = 1$

or $k^2n^2 (a^2 + b^2) = 1$

or $k = \frac{1}{n\sqrt{a^2 + b^2}}$

$\therefore \sin nt = \frac{b}{\sqrt{a^2 + b^2}}$, $\cos nt = \frac{a}{\sqrt{a^2 + b^2}}$

$\therefore$ amplitude $= a \cdot \frac{a}{\sqrt{a^2 + b^2}} = b \cdot \frac{b}{\sqrt{a^2 + b^2}}$

$$= \frac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2}$$

Example 2 : The velocity of a particle moving on a straight line at any time is given by $v^2 = 40x - 4x^2 - 64$. Show that the motion is simple harmonic.

Solution : We have $v^2 = 40x - 4x^2 - 64$

Differentiating both sides with respect to x, we get.

$$2v\frac{d^2v}{dx} = 40 - 8x$$

or $v\frac{dv}{dx} = 20 - 4x$

or $\frac{d^2x}{dt^2} = -4(x - 5)$

Taking x – 5 = y, we get

$$\frac{d^2y}{dt^2} = -4y \qquad (1)$$

From equation (1), we see that the acceleration is proportional to the distance and it is directed towards the fixed point. Therefore, the motion executed by the particle is simple harmonic.

Example 3 : A particle is moving with simple harmonic motion. At the end of three successive seconds, the distance of the particle from its mean position, measured in the same direction are 1, 5 and 5. Show that the period of a complete oscillation is $\frac{2p}{q}$, where $\cos q = \frac{3}{5}$.

Solution : We have $x = a\cos\sqrt{m}\,t$ (1)

Now, considering the conditions of question, we get

$1 = a\cos\sqrt{m}\,t$ (2)

$5 = a\cos\sqrt{m}\,(t+1)$ (3)

$5 = a\cos\sqrt{m}\,(t+2)$ (4)

Adding (2) and (4), we get

$$1 + 5 = a[\cos\sqrt{m}\,t + \cos\sqrt{m}\,(t+2)]$$

$$= 2a\cos\frac{\sqrt{m}\,(t+2)+\sqrt{m}\,t}{2}.\ \cos\frac{\sqrt{m}\,(t+2)-\sqrt{m}\,t}{2}$$

$$= 2a\cos\sqrt{m}\,(t+1)\cos\sqrt{m}$$

or $6 = 2\times 5\cos\sqrt{m}$, using (3)

$$\therefore\ \cos\sqrt{m} = \frac{6}{2\times 5} = \frac{3}{5}$$

$$\therefore\ \sqrt{m} = \cos^{-1}\frac{3}{5} = q,\ \text{(say)}$$

$$\therefore\ \text{periodic time} = \frac{2p}{\sqrt{m}} = \frac{2p}{q}\ \text{where}\ \cos q = \frac{3}{5}$$

Example 4: The speed n of a point P which moves in a straight line is given by the relation $v^2 = a - bx^2$, where x is the distance of the point P from a fixed point on the path, a and b being constants. Show that the motion of P is simple narmonic and determine its amplitude and period.

Solution : Given that $v^2 = a - bx^2$.

Differentiating with respect to x, we get

$$2v\frac{dv}{dx} = -2bx$$

or $$\frac{vdv}{dx} = -bx$$

or $$\frac{d^2v}{dt^2} = -bx \qquad \text{(i)} \qquad \left[\mathbf{Q}\,\frac{ndn}{dx} = \frac{d^2x}{dt^2}\right]$$

This equation is the hundard form of S.H.M. Hence the motion of the given particle is simple hermonic and its period $= \frac{2p}{b}$ $[m = b]$.

Also from $v^2 = a - bx^2$ when $n = 0$, we have

$$a - bx^2 = 0$$

or $x = \sqrt{\frac{a}{b}}$

We know that the amplitude of the motion is the value of x when n is zero, provided accolaration is zero at $n = 0$.

$\therefore$ amplitude, of the motion $= \sqrt{\frac{a}{b}}$



CHECK YOUR PROGRESS

Q 1: A particle is moving on a straight line and its velocity at a distance x from the origin is $l\sqrt{a^2 - x^2}$. Show that the motion is simple harmonic.

Q 2: The velocity of a particle moving on a straight line at any time is given by $v^2 = l^2(-x^2 + 10ax - 16a^2)$, where a and l are positive constants. Show that the motion is simple harmonic.

Q 3: A particle is moving on a straight line with simple harmonic motion. At the end of three successive seconds, the distances of the particle from its mean position, measured in the same direction are a, b and c respectively. Show that the period of a complete oscillation is $\frac{2p}{\cos^{-1}\frac{a+c}{2b}}$

Q 4: If V and f be the velocity and acceleration at any instant in a simple harmonic motion and T be the periodic time, show that $f^2T^2 + 4p^2V^2$ is a constant.

Q 5: A particle moving on a straight line with simple harmonic motion has velocities $v_1,\ v_2$ at distances $x_1,\ x_2$ respectively from the centre of the path. Show that the period of motion is $2p\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$ and the amplitude of motion is $\sqrt{\frac{v_2^2 x_1^2 - v_1^2 x_2^2}{v_2^2 - v_1^2}}$

Q 6: At what distance from the centre of oscillation will the velocity in a simple harmonic motion be half of the maximum?

Q 7: A particle is executing simple harmonic motion. Show that the particle requires one sixth of its period to move from the position of maximum displacement to one in which the displacement is half the amplitude.

Q 8: A body moving on a straight line OAB has its velocity zero when at points A and B whose distance from O are a and b respectively, and has velocity n when half-way between them. Show that the complete period is $\frac{p(b-a)}{v}$.

Q 9: A particle moving under simple harmonic motion on a straight line has velocities n_1, n_2 and accelerations $f_1,\ f_2$ in two of its positions on the path. If d is the distance between the two positions, show that $d = \frac{v_1^2 \sim v_2^2}{f_1 + f_2}$.

Q 10: A particle moving with S.H.M. has a velocity of 8m/s when it is at a distance of 3m from the centre of oscillation, and has a velocity of 6m/s when it is at a distance of 4m from the centre. Find its maximum velocity and periodic time.

6.3.1 Geometrical Representation of Simple Harmonic Motion

If a particle describes a circle with constant angular velocity, then the foot of the perpendicular from the particle on a diameter moves ander Simple Harmonic Motion. This is known as the geometrical respresentation of S.H.M.

Let a particle P describe a circle round O as centre and constant angular velocity w.

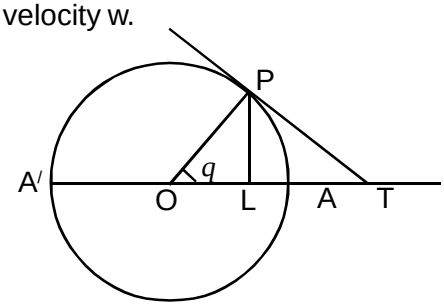


Fig 6.2

Let L be the foot of the perpendicular drawn from P on the fixed diameter A/OA.

When P is at A, L is also at A. As P moves from A towards A/ along the circle, L moves from A to A/ along the diameter AOA/. Finally, when P is at A/, L is also at A/. When P moves further from A/ towards A along the circle, L retraces its path from A/ to A along the diameter A/OA. Thus, the particle P oscillates from A to A/ and back.

Let a be the radius of the circle and P be the point (x, y) with O as origin and A/OA as the x-axis.

$\therefore\ x = a\cos q$

Now, the only acceleration of P towards O along the normal PO is w^2a.

The component of this acceleration along OA is

$w^2a\cos q = -w^2x$ towards O.

Thus, the equation of motion of the point L is

$$\frac{d^2x}{dt^2} = -w^2x$$

which is Simple Harmonic Motion.

Clearly, Periodic time $= \frac{2p}{\sqrt{w^2}} = \frac{2p}{w}$.

6.3.2 Motion of a Simple Pendulum

A heavy small particle suspended from a fixed point by a light inextensible string, and made to oscillate in a vertical plane, is called a simple pendulum. For small oscillation the motion is approximately rectilinear.

Let C be the fixed point. Let a heavy small particle be suspended from C by a light inextensible string. Let the length of the string be l.

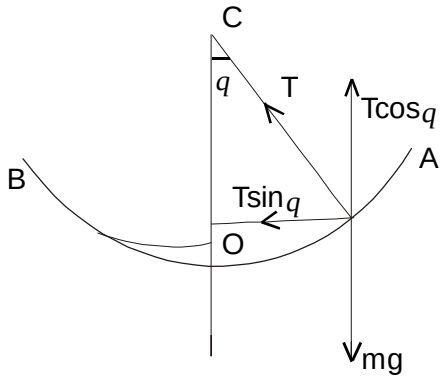


Fig 6.3

Let m be the mass of the particle.

Let the position of the particle at any time t be at P. when the particle is at P, let the string make an angle q with the vertical, where q is a very small angle.

Let PO be the horizontal displacement of the particle from its mean position O (the displacement is considered to be horizontal). Let OP = x.

Let T be the tension of the string.

Resolving horizontally and vertically, we have

$$T\cos q = mg \qquad (1)$$

$$-T\sin q = m\frac{d^2x}{dt^2} \qquad (2)$$

Since q is very small, $\therefore$ we take $\cos q = 1$

$\therefore$ from (1), T = mg

$\therefore$ from (2) $-mg\sin q = m\dfrac{d^2x}{dt^2}$

or, $\dfrac{d^2x}{dt^2} = -g\sin q$ (3)

Now from the right angled triangle COP,

$$\sin q = \frac{OP}{CP} = \frac{x}{l}$$

$\therefore$ from (3), $\dfrac{d^2x}{dt^2} = -\dfrac{g}{l}x$ (4)

From equation (4), we see that the acceleration is proportional to the distance and it is directed towards O. Therefore, the motion executed by the particle is simple harmonic.

Periodic time $= \dfrac{2p}{\sqrt{m}} = \dfrac{2p}{\sqrt{\dfrac{g}{l}}} = 2p\sqrt{\dfrac{l}{g}}$



6.4 LET US SUM UP

- If a particle moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance from the point, them the particle is said to execute Simple Harmonic Motion (S.H.M.).
- The velocity of a moving particle under S.H.M. at any distance x is $-\sqrt{m}\sqrt{a^2-x^2}$.
- The distance of a moving particle under S.H.M. from the fixed point at any time t is a cos $a\cos\sqrt{m}\,t$.
- The maximum displacement traversed by a particle under S.H.M. on either side of the centre of oscillation is called the amplitude.
- The time for one complete oscillation is called the periodic time or

the time of a complete oscillation. The periodic time is $\frac{2p}{\sqrt{m}}$, which is independent of the amplitude a.

- The number of complete oscillations per unit time is called the frequency of oscillation.
- If a particle describes a circle with constant angular velocity, then the foot of the perpendicular from the particle on a diameter moves under Simple Harmonic Motion. This is known as the geometrical representation of S.H.M.
- The motion of a simple pendulum is simple harmonic and its periodic time is $2p\sqrt{\frac{l}{g}}$, where l is the length of the pendulum and g, the acceleration due to gravity.



6.5 FURTHER READINGS

1) Statics and Dynamics with Background Mathematics, Adrian Roberts, Cambridge University Press.

2) Enginering Machanics : Statics and Dynamics by R. G. Hibbeler and Ashok Gupta.

3) Engineering Mechanics : Statics and Dynamics by S. Rajasekarn and G.Sankara subramanian. Vikas Publishing House Pvt. Ltd.



6.6 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: We have $v = l\sqrt{a^2 - x^2}$

or $v^2 = l^2\sqrt{a^2 - x^2}$

Differentiating both sides with respect to x, we get

$$2v\frac{dv}{dx} = l^2(0-2x)$$

or $v\frac{dv}{dx} = -l^2x$

or $\frac{d^2x}{dt^2} = -l^2x$ (1)

From equation (1), we see that the acceleration is proportional to the distance and it is directed towards the fixed point. Therefore, the motion executed by the particle is simple harmonic.

Ans to Q No 2: We have $v^2 = l^2(-x^2+10ax-16a^2)$

Differentiating both sides with respect to x, we get

$$2v\frac{dv}{dx} = l^2(-2x+10a)$$

or, $v\frac{dv}{dx} = -l^2(x-5a)$

or, $\frac{d^2x}{dt^2} = -l^2(x-5a)$

Taking x – 5a = y, we get

$$\frac{d^2y}{dt^2} = -l^2y \qquad (1)$$

From equation (1), we see that the acceleration is proportional to the distance and it is directed towards the fixed point. Therefore, the motion executed by the particle is simple harmonic.

Ans to Q No 3: We have $x = a\cos\sqrt{m}\,t$ (1)

Now, considering the conditions of question, we get

$a = a\cos\sqrt{m}\,t$ (2)

$b = a\cos\sqrt{m}\,(t+1)$ (3)

$c = a\cos\sqrt{m}\,(t+2)$ (4)

Adding (2) and (4), we get

$$a+c = a[\cos\sqrt{m}\,t + \cos\sqrt{m}\,(t+2)]$$

$$= 2a\cos\frac{\sqrt{m}\,(t+2)+\sqrt{m}\,t}{2}\cos\frac{\sqrt{m}\,(t+2)-\sqrt{m}\,t}{2}$$

$$= 2a\cos\sqrt{m}\,(t+1)\cos\sqrt{m}$$

$= 2b\cos\sqrt{m}$, using (3)

$\therefore \cos\sqrt{m} = \dfrac{a+c}{2b}$

or, $\sqrt{m} = \cos^{-1}\dfrac{a+c}{2b}$

$\therefore$ periodic time $= \dfrac{2p}{\sqrt{m}} = \dfrac{2p}{\cos^{-1}\dfrac{a+c}{2b}}$

Ans to Q No 4: We have $V = -\sqrt{m}\sqrt{a^2 - x^2}$

or, $V^2 = m\left(a^2 - x^2\right)$ (1)

$f = -m\,x$

or, $f^2 = m^2 x^2$

$T = \dfrac{2p}{\sqrt{m}}$

or, $T^2 = \dfrac{4p^2}{m}$

Now, $f^2T^2 + 4p^2v^2$

$= m^2x^2.\dfrac{4p^2}{m} + 4p^2m\left(a^2 - x^2\right)$

$= 4p^2m\,x^2 + 4p^2m\,a^2 - 4p^2m\,x^2$

$= 4p^2m\,a^2$ which is a constant.

Ans to Q No 5: We have $v_1^2 = m\left(a^2 - x_1^2\right)$ (1)

$v_2^2 = m\left(a^2 - x_2^2\right)$ (2)

$\therefore v_2^2 - v_1^2 = m\left(a^2 - x_2^2\right) - m\left(a^2 - x_1^2\right)$

$= m\left(x_1^2 - x_2^2\right)$

$\therefore m = \dfrac{v_2^2 - v_1^2}{x_1^2 - x_1^2}$

$\therefore$ periodic time $= \dfrac{2p}{\sqrt{m}} = 2p\sqrt{\dfrac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$

2nd Part Multiplying (1) by x_2^2 and (2) by x_1^2, We get

$v_1^2x_2^2 = m\,a^2\,x_2^2 - m\,x_1^2x_2^2$ (3)

$v_2^2 x_1^2 = m a^2 x_1^2 - m x_2^2 x_1$ (4)

Subtracting (3) from (4), we get

$m a^2 \left(x_1^2 - x_2^2\right) = v_2^2 x_1^2 - v_1^2 x_2^2$

or, $a^2 = \dfrac{v_2^2 x_1^2 - v_1^2 x_2^2}{m\left(x_1^2 - x_2^2\right)}$

or, $a = \dfrac{\sqrt{v_2^2 x_1^2 - v_1^2 x_2^2}}{\sqrt{m}\ \sqrt{x_1^2 - x_2^2}}$

$= \dfrac{\sqrt{v_2^2 x_1^2 - v_1^2 x_2^2}}{\sqrt{x_1^2 - x_2^2}} \cdot \dfrac{\sqrt{x_1^2 - x_2^2}}{\sqrt{v_2^2 - v_1^2}}$ using

$= \dfrac{\sqrt{v_2^2 x_1^2 - v_1^2 x_2^2}}{\sqrt{v_2^2 - v_1^2}}$

Ans to Q No 6: We have $v = \sqrt{m}\sqrt{a^2 - x^2}$

Considering the condition of question,

$v = \dfrac{1}{2}$ (maximum velocity)

or, $\sqrt{m}\sqrt{a^2 - x^2} = \dfrac{1}{2}\sqrt{m}\ a$

or, $\sqrt{a^2 - x^2} = \dfrac{1}{2} a$

or, $a^2 - x^2 = \dfrac{a^2}{4}$

or, $x^2 = a^2 - \dfrac{a^2}{4}$

or, $x^2 = \dfrac{3a^2}{4}$

$\therefore x = \dfrac{\sqrt{3}\, a}{2}$

Ans to Q No 7: We have $x = a\cos\sqrt{m}\, t$

Considering the condition of question, we get $x = \dfrac{1}{2} a$

or, $a\cos\sqrt{m}\,t=\frac{1}{2}a$

or, $\cos\sqrt{m}\,t=\frac{1}{2}$

or, $\sqrt{m}\,t=\frac{p}{3}$

or, $t=\frac{p}{3\sqrt{m}}$

$\therefore t=\frac{1}{6}\cdot\frac{2p}{\sqrt{m}}=\frac{1}{6}$ (periodic time)

Ans to Q No 8: We have velocity at A = 0

Velocity at B = 0

$\therefore$ A and B are the position of instantaneous rest.

Now AB = OB – OA

= b – a

$\therefore$ amplitude $=\frac{1}{2}AB=\frac{1}{2}(b-a)$

Let $\frac{1}{2}(b-a)=a$

We have $V^2=m\left(a^2-x^2\right)$

When x = 0, $V=v$

$\therefore v^2=ma^2=\frac{m}{4}(b-a)^2$

$\therefore m=\frac{4v^2}{(b-a)^2}$, or $\sqrt{m}=\frac{2v}{b-a}$

$\therefore$ periodic time $=\frac{2p}{\sqrt{m}}=\frac{2p(b-a)}{2v}$

$=\frac{p(b-a)}{v}$

Ans to Q No 9: We have $\frac{d^2x}{dt^2}=-m\,x$

and $v^2=m\left(a^2-x^2\right)$

Let the distances of the two positions from the centre of oscillation be x_1, x_2 respectively. We assume that $x_2 > x_1$

$\therefore\ f_1 = -m x_1,\ f_2 = -m x_2$ (1)

$n_1^{\,2} = m\left(a^2 - x_1^{\,2}\right),\ v_2^{\,2} = m\left(a^2 - x_2^{\,2}\right)$ (2)

$$\therefore v_1^{\,2} - v_2^{\,2} = m\left(a^2 - x_1^{\,2}\right) - m\left(a^2 - x_2^{\,2}\right)$$
$$= m\left(a^2 - x_1^{\,2} - a^2 - x_2^{\,2}\right)$$
$$= m\left(x_2^{\,2} - x_1^{\,2}\right)$$
$$= m\left(x_2 - x_1\right)\left(x_2 + x_1\right)$$
$$= m\,d\left(x_2 + x_1\right) \quad (3)$$

From (1) $f_1 + f_2 = -m\left(x_1 + x_2\right)$

or, $m = -\dfrac{f_1 + f_2}{x_1 + x_2}$ (4)

From (3) and (4),

$$v_1^{\,2} - v_2^{\,2} = -\frac{f_1 + f_2}{x_1 + x_2}\, d\left(x_1 + x_2\right)$$

or $d = -\dfrac{n_2^{\,2} - n_1^{\,2}}{f_1 + f_2}$

If we assume $x_1 > x_2$, $d = \dfrac{v_1^{\,2} - v_2^{\,2}}{f_1 + f_2}$

Thus, $d = \dfrac{v_1^{\,2} \sim v_2^{\,2}}{f_1 + f_2}$

Ans to Q No 10: We have $n^2 = m\left(a^2 - x^2\right)$ (1)

When x = 3, v = 8

from (1) $64 = m\left(a^2 - 9\right)$ (2)

When x = 4, $v = 6$

$\therefore$ from (1) $36 = m\left(a^2 - 16\right)$ (3)

Dividing (3) by (2), we get

$$\frac{36}{64} = \frac{a^2 - 16}{a^2 - 9}$$

or, $\dfrac{9}{16} = \dfrac{a^2 - 16}{a^2 - 9}$

or, $16a^2 - 256 = 9a^2 - 81$

or, $7a^2 = 175$

or, $7a^2 = 175$

or, $a^2 = 25$

or, $a = 5$.

$\therefore$ from (2), $64 = m(25-9)$

or $16m = 64$

$\therefore m = 4$

$\therefore$ Maximum velocity $= \sqrt{m}\, a$

= 2 x 5 m/s = 10 m/s

Periodic time $= \frac{2p}{\sqrt{m}} = \frac{2p}{2} = p$



6.7 MODEL QUESTIONS

Q 1: A particle executing simple harmonic motion has the velocities 10m/sec and 8m/sec at distances 6 m and 8m respectively from the centre of oscillation. Find the periodic time and the amplitude of the motion.

Q 2: A particle is moving under simple harmonic motion and its amplitude is a and the periodic time is T. Show that the maximum velocity of the particle is $\frac{2p\,a}{T}$.

Q 3: A particle moving under simple harmonic motion in a straight line, has velocities v_1, v_2, v_3 respectively at distances x_1, x_2, x_3, from the centre of oscillation. Prove that

$$x_1^2(v_2^2 - v_3^2) + x_2^2(v_3^2 - v_1^2) + x_3^2(v_1^2 - v_2^2) = 0$$

Q 4: A particle is executing simple harmonic motion with initial conditions $x = x_0$ and $v = v_0$ at t = 0. Show the amplitude is $\left(x_0^2 + \frac{v_0^2}{n^2}\right)^{\frac{1}{2}}$

Q 5: A particle is executing simple harmonic motion. It the velocities at distances a, b, c from a fixed point which is not the centre of oscillation are u, v, w respectively, show that the periodic time T is given by

$$\frac{4p^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ u^2 & v^2 & w^2 \end{vmatrix}.$$

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